ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 8, Number 3, Summer 1978

A NOTE ON EXCHANGEABLE SEQUENCES OF EVENTS

ERIC V. SLUD

ABSTRACT. Bruno de Finetti's (1931) representation for the law of an exchangeable sequence of 0's and 1's is exhibited as an invariant limit in the ergodic theorem for a transformation first defined by L. K. Arnold (1968) and studied by Hajian, Ito, and Kakutani (1972) in the context of σ-finite invariant measures. A known result on almost-sure convergence of normalized sums for such a sequence emerges as a corollary.

0. **Background.** A random sequence $\{X_k(\omega)\}_{k=1}^{\infty}$ on a probability space $(\Omega, \mathcal{F}, \mu)$ is said to be *exchangeable* if for any permutation σ of a finite set $\{k_1, \cdots, k_n\}$ of indices and for any $A_1, \cdots, A_n \in \mathcal{F}$ measurable events in Ω,

$$\mu(\{\omega \in \Omega : X_{\sigma(k_1)}(\omega) \in A_1, \cdots, X_{\sigma(k_n)}(\omega) \in A_n\}) = \mu(X_{k_1} \in A_1, \cdots, X_{k_n} \in A_n),$$

where we suppress explicit mention of ω on the right-hand side. A sequence of measurable events $\{A_k\}_{k=1}^{\infty}$ is called exchangeable whenever the sequence $\{I_{A_k}(\cdot)\}$ of its indicator functions is.

The simplest example of exchangeable events is that of an independent sequence $\{A_k\}_{k=1}^{\infty}$ with all $\mu(A_k)$ equal. This case occurs when the indicator $I_{A_k}(\omega) = \omega_k$ is the k^{th} coordinate of a point $\omega \in \{0, 1\}^{\infty} \cong \Omega$, where $\mathcal{F}$ is the product σ-algebra and μ the infinite-product probability measure on $\{0, 1\}^{\infty}$ assigning probability $\mu(A_1) = \mu(A_k)$ to $\{1\} \times \{0, 1\}^{\infty}$.

Given any sequence $\{A_k\}_{k=1}^{\infty}$ of measurable events in Ω, we can identify the measure spaces $(\Omega, \mathcal{F})$ and $\{0, 1\}^{\infty}$ via $\omega \to \{I_{A_k}(\omega)\}_{k=1}^{\infty}$.

So from now on we take $\Omega = \{0, 1\}^{\infty}$ with product σ-algebra $\mathcal{F}$, so that μ is the probability law of the random sequence $\omega = (\omega_1, \omega_2, \cdots) \in \{0, 1\}^{\infty}$. We assume that $\{X_k(\omega)\}_{k=1}^{\infty} \equiv \{I_{A_k}(\omega)\}_{k=1}^{\infty} = \{\omega_k\}_{k=1}^{\infty}$ is exchangeable, and call the measure μ exchangeable as well.

A celebrated theorem of Bruno de Finetti [2] says that the most general exchangeable measure μ on $\{0, 1\}^{\infty}$ is a mixture of infinite-product measures. There are many ways to prove this, including a particularly elementary combinatorial one due to Feller [3, p. 228]. In this paper we give a proof intended to shed immediate light on a further analogy between exchangeable and independent sequences:

Received by the editors on March 31, 1976, and in revised form on July 1, 1976.