On the Merger of Two Companies

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Abstract

This paper examines the merger of two stock companies under the premise, due to Bruno de Finetti, that the companies pay out dividends to their shareholders in such a way so as to maximize the expectation of the discounted dividends until (possible) ruin or insolvency. The aggregate net income streams of the two companies are modeled by a bivariate Wiener process. Explicit results are presented. In particular, it is shown that if for each company the product of the valuation force of interest and the square of the coefficient of variation of its aggregate net income process is less than 6.87%, the merger of the two companies would result in a gain.

1. Introduction

This paper is an application of actuarial risk theory. We study the question whether the merger of two stock companies is profitable and the amount of the resulting gain or loss. The basic premise, due to the Italian actuary Bruno de Finetti (1957), is that a company will pay out dividends to its shareholders in such a way so as to maximize the expectation of the discounted dividends. The dividend payments stop when the company becomes insolvent or is ruined. The economic value of the company is the expectation of the discounted dividends paid according to the *optimal* dividend strategy.

The aggregate net income streams of the two companies are modeled as a bivariate Wiener process (bivariate Brownian motion). For such companies, the optimal dividend strategy is a barrier strategy. Explicit results about the potential gain or loss upon merging the two companies are given. In particular, it is shown that, if for each company the product of the valuation force of interest and the square of the coefficient of variation of its aggregate net income process is less than 6.87%, the merger of the two companies would result in a gain. This conclusion is valid independently of the correlation between the two income streams.

The results are extended to the case where the capital earns interest income at a positive, constant rate.

2. The Model

For the convenience of the reader, this section presents some well-known results (see, e.g., Gerber 1972, Jeanblanc-Picqué and Shiryaev 1995, or Gerber and Shiu 2004) that will be used in the sequel. It is assumed that the *aggregate net income process* (before dividend payments) of a company is a Wiener process with a positive drift μ and variance σ^2 per unit time. Then the optimal dividend strategy is a *barrier strategy*. A barrier strategy has a parameter b, the level of the barrier. If the *capital* (also called *equity* or *surplus*) of the company is less than b, no dividends are paid. Whenever the capital reaches the level b, the "overflow" is paid as dividends to the shareholders. If the initial capital exceeds b, the difference is paid immediately as dividends. The current capital is the sum of the initial capital and aggregate net income, less distributed dividends. Ruin or insolvency occurs as soon as the capital becomes negative.

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Let $V(x; b)$ denote the expectation of the discounted dividends until ruin, if the barrier strategy corresponding to parameter b is applied. Here, x denotes the initial capital. Then

$$V(x; b) = \frac{e^{rx} - e^{sx}}{re^{rb} - se^{sb}} \qquad \text{for } 0 \leq x \leq b, \tag{2.1}$$

with

$$r = \frac{-\mu + \sqrt{\mu^2 + 2\delta\sigma^2}}{\sigma^2}, \tag{2.2a}$$

$$s = \frac{-\mu - \sqrt{\mu^2 + 2\delta\sigma^2}}{\sigma^2}, \tag{2.2b}$$

where $\delta > 0$ is the valuation force of interest. Furthermore,

$$V(x; b) = x - b + V(b; b) \qquad \text{for } x > b, \tag{2.3}$$

because in this case, the initial excess is paid immediately as a dividend to the stockholders.

Let b^* denote the optimal value of b; that is, b^* is the value of b that minimizes the denominator in (2.1). Setting the derivative of the denominator in (2.1) equal to zero, we have

$$r^2 e^{rb^*} - s^2 e^{sb^*} = 0. \tag{2.4}$$

Hence,

$$b^* = \frac{1}{r - s} \ln\left(\frac{s^2}{r^2}\right) = \frac{2}{r - s} \ln\left(\frac{-s}{r}\right). \tag{2.5}$$

The *Dividend Discount Model* (also known as the *Gordon Model*) in finance determines the price of a stock by discounting its projected or expected dividend payments (Williams 1938; Gordon 1962). See also the actuarial monographs Booth et al. (2005, Section 2.6.1) and Toole and Herget (2005, Section 4.2.5). Under this model, the (economic) value of the company with capital x is $V(x; b^*)$.

From (2.4) and (2.1), it follows that

$$V''(b^*; b^*) = 0, \tag{2.6}$$

from which and (2.3) we see that the second derivative of the function $V(x; b^*)$ is continuous at $x = b^*$, which is a *smooth pasting condition*. Furthermore,

$$V(b^*; b^*) = \frac{\mu}{\delta}. \tag{2.7}$$

This elegant formula will be a key for analyzing a merger.

The shape of the value function $V(x; b^*)$, $x > 0$, can be readily obtained by noting that $V''(x; b^*) < 0$ and $V'(x; b^*) > 1$ for $0 < x < b^*$, and $V'(x; b^*) = 1$ for $x \geq b^*$ (see Fig. 1).

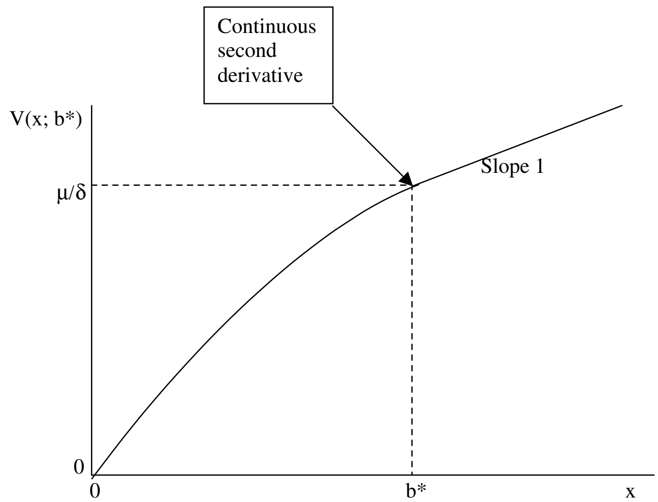
3. AN ALTERNATIVE EXPRESSION FOR THE OPTIMAL BARRIER

For further discussion, it is judicious to express the optimal barrier b^* more directly in terms of the parameters of the model. Let

$$\zeta = \frac{\sigma}{\mu} \tag{3.1}$$

be the *coefficient of variation* of the underlying Wiener process. Then, it follows from (2.5), (2.2a), and (2.2b) that

Figure 1
The Value of the Company as a Function of Its Capital



$$b^* = \mu f(\zeta), \tag{3.2}$$

where

$$f(z) = \frac{z^2}{\sqrt{1 + 2\delta z^2}} \ln\left(\frac{\sqrt{1 + 2\delta z^2} + 1}{\sqrt{1 + 2\delta z^2} - 1}\right), \qquad z \geq 0. \tag{3.3}$$

As the function $f(z)$ hinges on the parameter δ, it is useful to introduce the parameter-free function

$$g(y) = \frac{y^2}{\sqrt{1 + y^2}} \ln\left(\frac{\sqrt{1 + y^2} + 1}{\sqrt{1 + y^2} - 1}\right), \qquad y \geq 0. \tag{3.4}$$

Then

$$f(z) = \frac{1}{2\delta} g(\sqrt{2\delta}\, z), \qquad z \geq 0, \tag{3.5}$$

and

$$b^* = \frac{\mu}{2\delta} g(\sqrt{2\delta}\, \zeta). \tag{3.6}$$

The function $g(y)$, $y \geq 0$, is an increasing function, with $g(0) = 0$ and $g(\infty) = 2$. We shall be interested in the convexity of f, hence in the convexity of g. This property can be readily examined by mathematical software such as Maple or Mathematica. It is found that there exists a number $\tilde{y} = 0.3708175 \ldots$ such that

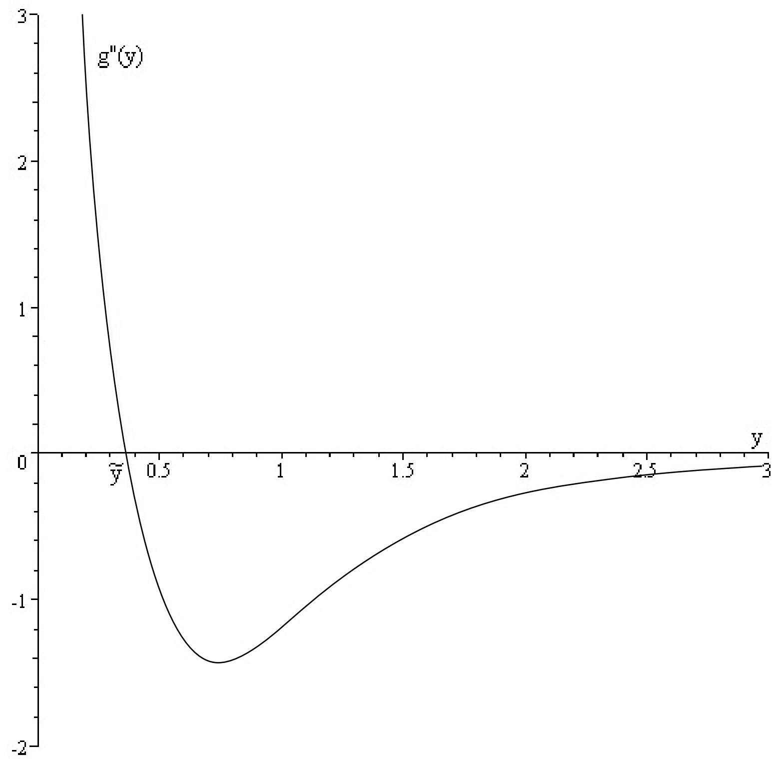
$$g''(y) > 0 \quad \text{for } 0 \leq y < \tilde{y}, \tag{3.7a}$$

and

$$g''(y) < 0 \quad \text{for } \tilde{y} < y < \infty. \tag{3.7b}$$

The graph of the second derivative $g''(y)$ is displayed in Figure 2. The zero of g'' is $\tilde{y}$.

Figure 2
Second Derivative $g''(y)$



4. The Situation before and after the Merger

We consider two stock companies, labeled 1 and 2, and assume that the aggregate net income process (before dividend payments) of company j is a Wiener process with positive drift μ_j and variance σ_j^2 per unit time. The optimal barrier for company j is

$$b_j^* = \mu_j f(\zeta_j), \tag{4.1}$$

with f defined by (3.3) and

$$\zeta_j = \frac{\sigma_j}{\mu_j}. \tag{4.2}$$

Let $V_j(x; b)$ denote the expectation of the company's discounted dividends until ruin, if x is its capital and the barrier strategy corresponding to parameter b is applied.

Furthermore, we assume that the joint aggregate net income process (before dividend payments) is a bivariate Wiener process with correlation coefficient ρ. It is assumed that a merger does not affect the model and its parameters. Hence, after the merger, the resulting aggregate net income process (before dividend payments) is a Wiener process, with parameters

$$\mu_m = \mu_1 + \mu_2 \tag{4.3}$$

and

$$\sigma_m^2 = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2. \tag{4.4}$$

Thus, the optimal barrier for the merged company is

$$b_m^* = (\mu_1 + \mu_2)f(\zeta_m), \tag{4.5}$$

with

$$\zeta_m = \frac{\sigma_m}{\mu_m} = \frac{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}}{\mu_1 + \mu_2}. \tag{4.6}$$

Let $V_m(x; b)$ denote the expectation of the merged company's discounted dividends until ruin. Then the potential gain on merging the two companies is

$$V_m(x_1 + x_2; b_m^*) - [V_1(x_1; b_1^*) + V_2(x_2; b_2^*)], \tag{4.7}$$

where x_j is the current capital of company j, $j = 1, 2$, and $x_1 + x_2$ is the capital of the merged company. Each of the three terms in (4.7) can be calculated by applying the appropriate version of (2.1), (2.2), and (2.5). A merger is profitable if expression (4.7) is positive.

There is a situation where the sign of expression (4.7) can be readily identified. It follows from (2.7) and (4.3) that

$$V_m(b_m^*; b_m^*) = V_1(b_1^*; b_1^*) + V_2(b_2^*; b_2^*). \tag{4.8}$$

Therefore, let us assume that the current capital of company j is b_j^*, $j = 1, 2$. Hence, $(b_1^* + b_2^*)$ is the capital of the merged company. Because $V_m(x; b_m^*)$ is an increasing function of x, the following two conclusions are immediate consequences of (4.8):

(a) If

$$b_m^* < b_1^* + b_2^*, \tag{4.9}$$

then

$$V_1(b_1^*; b_1^*) + V_2(b_2^*; b_2^*) < V_m(b_1^* + b_2^*; b_m^*). \tag{4.10}$$

That is, the merger results in a gain.

(b) If

$$b_m^* > b_1^* + b_2^*, \tag{4.11}$$

then

$$V_1(b_1^*; b_1^*) + V_2(b_2^*; b_2^*) > V_m(b_1^* + b_2^*; b_m^*). \tag{4.12}$$

That is, a merger makes no economic sense.

Further conclusions about the gain or loss can be made by examining Figure 1 with $V = V_m$ and $b^* = b_m^*$. If (4.9) is satisfied, the gain due to merger consists of an immediate release of capital of amount

$$V_m(b_1^* + b_2^*; b_m^*) - V_m(b_m^*; b_m^*) = b_1^* + b_2^* - b_m^*.$$

If (4.11) is satisfied, a merger would result in a loss of value equal to

$$V_m(b_m^*; b_m^*) - V_m(b_1^* + b_2^*; b_m^*),$$

which is greater than $b_m^* - (b_1^* + b_2^*)$.

5. A SUFFICIENT CONDITION FOR MERGER

The purpose of this section is to derive a sufficient condition for (4.9) that does not depend on ρ. As a function of ρ, the optimal barrier b_m^*, given by (4.5), is an increasing function. To see this, note that the coefficient of variation ζ_m, given by (4.6), is an increasing function of ρ, and that the function f is an increasing function (because the function g is an increasing function). Hence, if (4.9) holds for $\rho = 1$, it holds for all $\rho \in [-1, 1]$.

We now examine the case $\rho = 1$. It follows from (4.6) that

$$\zeta_m = \frac{\sigma_1 + \sigma_2}{\mu_1 + \mu_2}, \tag{5.1}$$

which can be written as a weighted average of ζ_1 and ζ_2:

$$\zeta_m = \frac{\mu_1}{\mu_m}\zeta_1 + \frac{\mu_2}{\mu_m}\zeta_2. \tag{5.2}$$

Thus, if $\zeta_1 = \zeta_2$, then ζ_m has the same value, and $b_m^* = b_1^* + b_2^*$. If $\zeta_1 \neq \zeta_2$, then condition (4.9) is now the condition

$$f\left(\frac{\mu_1}{\mu_m}\zeta_1 + \frac{\mu_2}{\mu_m}\zeta_2\right) < \frac{\mu_1}{\mu_m}f(\zeta_1) + \frac{\mu_2}{\mu_m}f(\zeta_2) \tag{5.3}$$

because of (4.5) and (4.1). A sufficient condition for (5.3) to hold is that the function $f(z)$ is convex for z between ζ_1 and ζ_2, or that $f''(z) > 0$ for z between ζ_1 and ζ_2. Now we recall from (3.7a) that $g''(y) > 0$ for $0 \leq y < \tilde{y} = 0.3708175\ldots$ It thus follows from (3.5) that if

$$\sqrt{2\delta}\,\zeta_j < \tilde{y} \tag{5.4}$$

for $j = 1$ and 2, we can be sure that inequality (4.9) holds.

Consequently, if for both companies 1 and 2 we have

$$\frac{\delta\sigma_j^2}{\mu_j^2} < \frac{\tilde{y}^2}{2} = 0.0687528\ldots, \tag{5.5}$$

and if the current capital of each company is at its optimal dividend barrier, then a merger would be profitable. This result holds for every correlation coefficient ρ.

6. A Generalization

Cai, Gerber, and Yang (2006) enrich the model in Section 2 by assuming that the capital accumulates at a constant force of interest, which we denote by ι here, $0 < \iota < \delta$. Again, we let $V(x; b)$ denote the expectation of the discounted dividends until ruin, if the barrier strategy corresponding to parameter b is applied, and let b^* denote the optimal barrier. For $\iota > 0$, there is no known closed-form expression for b^*. Nevertheless, Cai, Gerber, and Yang show that

$$V(b^*; b^*) = \frac{\mu + \iota b^*}{\delta}, \tag{6.1}$$

extending (2.7). Furthermore, $V''(x; b^*) < 0$ and $V'(x; b^*) > 1$ for $0 < x < b^*$, and $V'(x; b^*) = 1$ for $x \geq b^*$. Hence, Figure 1 remains valid if we replace μ/δ on the vertical axis by $(\mu + \iota b^*)/\delta$. Thus, much of Section 4 can be adapted to the framework of the Cai, Gerber, and Yang (2006) model, as we now explain.

Again, we consider two stock companies, labeled 1 and 2. Let V_1 and V_2 be their expected present value of dividends functions, and b_1^* and b_2^* be the optimal barriers, respectively. For the merged company, let V_m be its expected present value of dividends function and b_m^* be the optimal barrier. It then follows from (6.1) and (4.3) that

$$V_m(b_m^*; b_m^*) = V_1(b_1^*; b_1^*) + V_2(b_2^*; b_2^*) + \frac{\iota}{\delta}(b_m^* - b_1^* - b_2^*), \tag{6.2}$$

which generalizes (4.8).

A merger makes economic sense if and only if condition (4.9) is satisfied. If (4.9) holds, the gain due to a merger is

$$\begin{aligned} &V_m(b_1^* + b_2^*; b_m^*) - [V_1(b_1^*; b_1^*) + V_2(b_2^*; b_2^*)] \\ &\quad = [V_m(b_1^* + b_2^*; b_m^*) - V_m(b_m^*; b_m^*)] + \{V_m(b_m^*; b_m^*) - [V_1(b_1^*; b_1^*) + V_2(b_2^*; b_2^*)]\} \\ &\quad = (b_1^* + b_2^* - b_m^*) + \frac{\iota}{\delta}(b_m^* - b_1^* - b_2^*) \\ &\quad = \left(1 - \frac{\iota}{\delta}\right)(b_1^* + b_2^* - b_m^*). \end{aligned} \tag{6.3}$$

On the other hand, if (4.11) holds, the loss due to a merger is

$$[V_1(b_1^*; b_1^*) + V_2(b_2^*; b_2^*)] - V_m(b_1^* + b_2^*; b_m^*),$$

which is greater than $(1 - \iota/\delta)(b_m^* - b_1^* - b_2^*)$ because, in view of the left half of Figure 1,

$$V_m(b_m^*; b_m^*) - V_m(b_1^* + b_2^*; b_m^*) > b_m^* - (b_1^* + b_2^*).$$

REMARK

In this model, where the capital earns investment interest at a positive rate, the barrier strategy with parameter b^* is optimal among all possible dividend strategies if $x \le b^*$. It is a conjecture (which has not been proven to our knowledge) that it is also optimal if $x > b^*$. Even if the conjecture were wrong, the results of this section would be of interest: one might argue that a priori only barrier strategies are admissible; this restriction agrees with the philosophy in Dickson and Waters (2004).

7. CONCLUDING REMARKS

We have tacitly assumed that the two companies as well as the merged company would discount their dividends with the same valuation force of interest δ. This assumption facilitates the analysis. In practice, the applicable valuation force of interest may need to be risk adjusted: that is, it is a function of the two parameters of the Wiener process. In other words, δ is to be replaced by δ_1, δ_2, and δ_m.

Also, it is usually expected that a merger will generate some synergy. In the model this would mean that

$$\mu_m \ge \mu_1 + \mu_2 \tag{7.1}$$

and

$$\sigma_m^2 \le \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2. \tag{7.2}$$

Because there are costs in each merger, formula (4.7) should be modified as

$$V_m(x_1 + x_2 - \alpha; b_m^*) - [V_1(x_1; b_1^*) + V_2(x_2; b_2^*)], \tag{7.3}$$

where α represents the cost of the merger. Each of the three terms in (7.3) can be readily evaluated using an appropriate version of (2.1), (2.2), and (2.5). If the quantity (7.3) is positive, then the merger is profitable.

This paper addressed the question "Merger—Now or Never?" and found a relatively simple answer. One might argue that if merger does not take place now, it remains an *option* in the future. This leads to the problem of finding the optimal time of a merger. This question has been pointed out to us by Dr. Mihael Perman and by an anonymous referee. It would be interesting to see whether this new and challenging problem has an attractive solution.

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