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Some Highlights from the Theory of Multivariate Symmetries

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Olav Kallenberg
Auburn University

Basic Symmetries

For infinite random sequences $X = (X_j)$, we have the following basic symmetries, listed in order of strength, along with the associated classes of transformations. (Thus, X has the property on the left iff its distribution is invariant under the transformations on the right.)

stationary	*shifts*
contractable	*contractions*
exchangeable	*permutations*
rotatable	*rotations*

In particular, X is *contractable* if all subsequences have the same distribution and *rotatable* if the joint distribution is invariant under any orthogonal transformation of finitely many elements.

Classical Results for Infinite Sequences

For infinite sequences $X = (X_j)$ of random variables, we have the following basic characterizations:

♣ **de Finetti (1937):** *X is exchangeable iff it is mixed (or conditionally) i.i.d.*

♣ **Ryll-Nardzewski (1957):** *X is contractable iff it is exchangeable, hence mixed i.i.d.*

♣ **Freedman (1962):** *X is rotatable iff it is mixed i.i.d. centered Gaussian*

Symmetries on Two-Dimensional Arrays

For random arrays of the form

$$X = (X_{ij};\ i, j \geq 1),$$

we define

$$(X \circ (p, q))_{ij} = X_{p_i, q_j}, \quad i, j \geq 1.$$

Say that X is

- *separately exchangeable* if $X \circ (p, q) \stackrel{d}{=} X$ for all permutations $p = (p_i)$ and $q = (q_j)$,
- *jointly exchangeable* if $X \circ (p, p) \stackrel{d}{=} X$ for all permutation $p = (p_i)$.

The definitions of separate or joint contractability or rotatability are similar. So are the definitions for higher-dimensional arrays. Note that separate exchangeability and contractability are equivalent.

Natural Index Sets

An array $X = (X_{ij})$ on $\mathbb{N}^2$ is *jointly exchangeable* iff the same property holds for the array

$$Y_{ij} = (X_{ij}, X_{ii}), \quad i \neq j.$$

It is then enough to consider arrays indexed by the *non-diagonal* part of $\mathbb{N}^2$. In general, we may consider exchangeable arrays on the class $\overline{\mathbb{N}}$ of *finite sequences of distinct numbers in* $\mathbb{N}$.

Similarly, an array $X = (X_{ij})$ on $\mathbb{N}$ is *jointly contractable* iff the same property holds for the array

$$Z_{ij} = (X_{ij}, X_{ji}, X_{ii}), \quad i < j.$$

It is then enough to consider arrays indexed by the *sub-diagonal* part of $\mathbb{N}^2$, which may be identified with the set of unordered pairs $\{i, j\}$. In general, we may consider contractable arrays on the class $\widetilde{\mathbb{N}}$ of *finite subsets* of $\mathbb{N}$.

Functional Form of de Finetti's Theorem

♣ *An infinite sequence $X = (X_n)$ of r.v.'s is exchangeable iff a.s.*

$$X_n = f(\alpha, \xi_n), \quad n \geq 1,$$

for a measurable function f on $[0,1]^2$ and some i.i.d. $U(0,1)$ r.v.'s α and $\xi_1, \xi_2, \ldots$.

Here the X_n are conditionally i.i.d. given α. The function f is not unique, and the construction of α and $\xi_1, \xi_2, \ldots$ may require an extension of the basic probability space.

Representation of Exchangeable Arrays

♣ **Aldous (1981), Hoover (1979):** *An array $X = (X_{ij})$ is separately exchangeable iff a.s.*

$$X_{ij} = f(\alpha, \xi_i, \eta_j, \zeta_{ij}), \quad i, j \geq 1,$$

for a measurable function f on $[0,1]^4$ and some i.i.d. $U(0,1)$ r.v.'s α, ξ_i, η_j, ζ_{ij}.

♣ **Hoover (1979):** *An array $X = (X_{ij};\, i \neq j)$ is jointly exchangeable iff a.s.*

$$X_{ij} = f(\alpha, \xi_i, \xi_j, \zeta_{ij}), \quad i \neq j,$$

for a measurable function $f : [0,1]^4 \to \mathbb{R}$ and some i.i.d. $U(0,1)$ r.v.'s α, ξ_i, $\zeta_{ij} = \zeta_{ji}$, $i \neq j$.

Representation and Extension of Contractable Arrays

♣ **K(1992):** *An array* $X = (X_{ij};\, i < j)$ *is jointly contractable iff a.s.*

$$X_{ij} = f(\alpha, \xi_i, \xi_j, \zeta_{ij}), \quad i < j,$$

for a measurable function f *on* $[0,1]^4$ *and some i.i.d.* $U(0,1)$ *r.v.'s* α, ξ_i, ζ_{ij}, $i < j$.

Comparing with the jointly exchangeable case gives:

♣ *An sub-diagonal array is jointly contractable iff it can be extended to a jointly exchangeable array on the non-diagonal index set.*

No direct proof is known.

Higher-Dimensional Arrays

For $k = (k_1, \ldots, k_n)$, let $\tilde{k} = \{k_1, \ldots, k_n\}$.

♣ **Hoover (1979):** *An array X on $\overline{\mathbb{N}}$ is jointly exchangeable iff a.s.*

$$X_k = f(\xi_I;\, I \subset \tilde{k}), \quad k \in \overline{\mathbb{N}},$$

for a measurable function f on $\bigcup_n [0,1]^{2^n}$ and some i.i.d. $U(0,1)$ r.v.'s ξ_I, $I \in \widetilde{\mathbb{N}}$.

♣ **K(1992):** *An array X on $\widetilde{\mathbb{N}}$ is jointly contractable iff a.s.*

$$X_J = f(\xi_I;\, I \subset J), \quad J \in \widetilde{\mathbb{N}},$$

for a measurable function f on $\bigcup_n [0,1]^{2^n}$ and some i.i.d. $U(0,1)$ r.v.'s ξ_I, $I \in \widetilde{\mathbb{N}}$.

♣ *An array on $\widetilde{\mathbb{N}}$ is jointly contractable iff it can be extended to an jointly exchangeable array on $\overline{\mathbb{N}}$.*

Rotatable Arrays

♣ **Freedman (1962):** *A sequence $X = (X_i)$ is rotatable iff a.s.*

$$X_i = \sigma\zeta_i, \quad i \geq 1,$$

for some i.i.d. $N(0,1)$ r.v.'s ζ_i and an independent r.v. σ.

♣ **Aldous (1981):** *An array $X = (X_{ij})$ is separately rotatable iff a.s.*

$$X_{ij} = \sigma\zeta_{ij} + \sum\nolimits_k \alpha_k \, \xi_{ki} \, \eta_{kj}, \quad i, j \geq 1,$$

for some i.i.d. $N(0,1)$ r.v.'s ξ_{ki}, η_{kj}, ζ_{ij} and an independent set of r.v.'s σ and α_k satisfying $\sum_k \alpha_k^2 < \infty$.

♣ **K(1988):** *An array $X = (X_{ij})$ is jointly rotatable iff a.s.*

$$\begin{aligned} X_{ij} &= \rho\delta_{ij} + \sigma\zeta_{ij} + \sigma'\zeta_{ji} \\ &\quad + \sum\nolimits_{h,k} \alpha_{hk} \, (\xi_{hi} \, \xi_{kj} - \delta_{ij}\delta_{hk}), \quad i, j \geq 1, \end{aligned}$$

for some i.i.d. $N(0,1)$ r.v.'s ξ_{hi}, ζ_{ij} and an independent set of r.v.'s ρ, σ, σ', α_{hk} satisfying $\sum_{h,k} \alpha_{hk}^2 < \infty$.

Hilbert-Space Setting

By a *continuous, linear, random functional (CLRF)* on a (separable, infinite-dimensional, real) Hilbert space H we mean a random process X on H such that

- $h_n \to 0$ in H implies $X h_n \xrightarrow{P} 0$,
- $X(ah + bk) = aXh + bXk$ a.s.
  for all $h, k \in H,\ a, b \in \mathbb{R}$.

An *isonormal Gaussian process (G-process)* on H is defined as a centered Gaussian process X on H such that

$$\mathrm{Cov}(Xh, Xk) = \langle h, k \rangle, \quad h, k \in H.$$

A *unitary operator* on H is a linear isometry U of H onto itself. Say that a CLRF X is *rotatable* if $X \circ U \overset{d}{=} X$ for all unitary operators U, where $(X \circ U)h = X(Uh)$.

♣ **Freedman (1962–63):** *A CLRF X on H is rotatable iff $X = \sigma\eta$ a.s. for some G-process η and an independent r.v. $\sigma \geq 0$.*

Multi-Variate Rotations

To understand tensor products, we may assume that $H_k = L^2(\mu_k)$ for all k. Then

$$H_1 \otimes \cdots \otimes H_n = L^2(\mu_1 \otimes \cdots \otimes \mu_n),$$
$$(h_1 \otimes \cdots \otimes h_n)(s_1, \ldots, s_n) = h_1(s_1) \cdots h_n(s_n).$$

For any unitary operators U_k on H_k, $k \leq n$, there exists a unique unitary operator $\bigotimes_k U_k = U_1 \otimes \cdots \otimes U_n$ on $\bigotimes_k H_k = H_1 \otimes \cdots \otimes H_n$ such that

$$(U_1 \otimes \cdots \otimes U_n)(h_1 \otimes \cdots \otimes h_n) = U_1 h_1 \otimes \cdots \otimes U_n h_n.$$

Write $H^{\otimes n} = H \otimes \cdots \otimes H$ and $U^{\otimes n} = U \otimes \cdots \otimes U$.

A CLRF X on $H^{\otimes n}$ is said to be

- *separately rotatable* if $X \circ \bigotimes_k U_k \stackrel{d}{=} X$ for all unitary operators $U_1, \ldots, U_n$ on H,
- *jointly rotatable* if $X \circ U^{\otimes n} \stackrel{d}{=} X$ for all unitary operators U on H.

Multiple Wiener–Itô Integrals

The basic examples of separately or jointly rotatable CLRF's are the *multiple WI-integrals*, defined as follows:

♣ *For any independent G-processes* η_k *on* H_k, $k \le n$, *there exists an a.s. unique CLRF* $\bigotimes_k \eta_k$ *on* $\bigotimes_k H_k$ *such that, a.s. for any* $h_k \in H_k$,

$$(\eta_1 \otimes \cdots \otimes \eta_n)(h_1 \otimes \cdots \otimes h_n) = \eta_1 h_1 \cdots \eta_n h_n.$$

♣ *For any G-process* η *on* H *and any* $n \in \mathbb{N}$, *there exists an a.s. unique CLRF* $\eta^{\otimes n}$ *on* $H^{\otimes n}$ *such that, a.s. for orthogonal* $h_1, \ldots, h_n \in H$,

$$\eta^{\otimes n}(h_1 \otimes \cdots \otimes h_n) = \eta h_1 \cdots \eta h_n.$$

Clearly $\bigotimes_k \eta_k$ is separately rotatable and $\eta^{\otimes n}$ is jointly rotatable. Similarly, we may define CLRF's on $\bigotimes_k H_k^{\otimes r_k}$ of the form $\bigotimes_k \eta_k^{\otimes r_k}$.

Separately Rotatable Random Functionals

Let $\mathcal{P}_d$ be the set of partitions of $\{1, \dots, d\}$. Put $H^{\otimes J} = \bigotimes_{j \in J} H$ and $H^{\otimes \pi} = \bigotimes_{J \in \pi} H$.

♣ **K(1995)** *A CLRF X on $H^{\otimes d}$ is separately rotatable iff a.s.*

$$Xf = \sum_{\pi \in \mathcal{P}_d} \Big(\bigotimes_{J \in \pi} \eta_J \Big)(\alpha_\pi \otimes f), \quad f \in H^{\otimes d},$$

for some independent G-processes η_J on $H \otimes H^{\otimes J}$, $J \in 2^d \setminus \{\emptyset\}$, and an independent set of random elements $\alpha_\pi \in H^{\otimes \pi}$, $\pi \in \mathcal{P}_d$.

This is equivalent to the basis representation

$$X_k = \sum_{\pi \in \mathcal{P}_d} \sum_{l \in \mathbb{N}^\pi} \alpha_l^\pi \prod_{J \in \pi} \eta_{k_J, l_J}^J, \quad k \in \mathbb{N}^d,$$

where

$$X_{k_1, \dots, k_d} = X(h_{k_1} \otimes \cdots \otimes h_{k_d})$$

for some ONB $h_1, h_2, \ldots \in H$. Any separately rotatable array can be represented in this form.

Jointly Rotatable Random Functionals

Let $\mathcal{O}_d$ be the class of partitions of $\{1, \ldots, d\}$ into *ordered* sets $k = (k_1, \ldots, k_r)$ of size $|k| = r$.

♣ **K(1995)** *A CLRF X on H^d is jointly rotatable iff a.s.*

$$Xf = \sum_{\pi \in \mathcal{O}_d} \big(\bigotimes_{k \in \pi} \eta_{|k|}\big)(\alpha_\pi \otimes f), \quad f \in H^{\otimes d},$$

for some independent G-processes η_k on $H^{\otimes(k+1)}$, $k \leq d$, and an independent set of elements $\alpha_\pi \in H^{\otimes \pi}$, $\pi \in \mathcal{O}_d$.

This may again be restated in basis form, using the representation of multiple WI-integrals in terms of Hermite polynomials. However, the general representation of jointly rotatable arrays is more complicated, since it also involves diagonal terms of different order.

Separately Exchangeable Random Sheets

A *random sheet* on $\mathbb{R}_+^d$ is a continuous process $X = (X_t)$ such that $X_t = 0$ when $\min_j t_j = 0$. Exchangeability and contractability are defined in terms of the increments.

Let $\hat{\mathcal{P}}_d = \bigcup_J \mathcal{P}_J$, where $\mathcal{P}_J$ is the class of partitions of $J \in 2^d \setminus \{\emptyset\}$. For $\pi \in \mathcal{P}_J$, write $\pi^c = J^c$.

♣ **K(1995)** *A random sheet X on $\mathbb{R}_+^d$ is separately exchangeable iff a.s.*

$$X_t = \sum_{\pi \in \hat{\mathcal{P}}_d} (\lambda^{\pi^c} \otimes \bigotimes_{J \in \pi} \eta_J)(\alpha_\pi \otimes [0,t]), \quad t \in \mathbb{R}_+^d,$$

for some independent G-processes η_J on $H \otimes L^2(\lambda^J)$ and an independent set of random elements $\alpha_\pi \in H^{\otimes \pi}$.

A similar representation holds for separately exchangeable random sheets on $[0,1]^d$.

Jointly Exchangeable and Contractable Random Sheets

For any $\pi \in \hat{\mathcal{P}}_d$, put $\hat{\mathcal{O}}_\pi = \bigcup_{J\in\pi} \mathcal{O}_J$, and define the vectors $\hat{t}_\pi$ by $\hat{t}_{\pi,J} = \min_{j\in J} t_j$, $J \in \pi$.

♣ **K(1995)** *A random sheet X on $\mathbb{R}_+^d$ is jointly exchangeable iff, a.s. for all $t \in \mathbb{R}_+^d$,*

$$X_t = \sum_{\pi\in\mathcal{P}_d} \sum_{\kappa\in\hat{\mathcal{O}}_\pi} (\lambda^{\pi^c} \otimes \bigotimes_{k\in\kappa} \eta_{|k|})(\alpha_{\pi,\kappa} \otimes [0, \hat{t}_\pi]),$$

for some independent G-processes η_m on $H \otimes L^2(\lambda^m)$, $m \le d$, and an independent set of random elements $\alpha_{\pi,\kappa} \in H^{\otimes\kappa}$, $\kappa \in \hat{\mathcal{O}}_\pi$, $\pi \in \mathcal{P}_d$.

A similar but more complicated representation holds for jointly contractable sheets on $\mathbb{R}_+^d$. The problem of characterizing jointly exchangeable sheets on $[0,1]^d$ remains open.

1930–37	*de Finetti*	*exchangeable sequences*
1938	*de Finetti*	*partial exchangeability*
1938	*Schoenberg*	*completely monotone functions*
1951	*Itô*	*multiple Wiener integrals*
1957	*Ryll-Nardzewski*	*contractable sequences*
1960	*Bühlmann*	*exchangeable processes*
1961	*Gaifman*	*exchangeable arrays in logic*
1962–63	*Freedman*	*rotatable sequences/processes*
1969	*Krauss*	*exchangeable arrays in logic*
1970–73	*Olson/Uppuluri*	*rotatable matrices*
1972–78	*Dawid*	*exchangeable/rotatable arrays*
1975	*McGinley/Sibson*	*exchangeable arrays*
1976	*Silverman*	*exchangeable arrays*
1978	*Eagleson/Weber*	*exchangeable arrays*
1979	*Hoover*	*exchangeable arrays*
1981	*Aldous*	*exchangeable/rotatable arrays*
1981	*Diaconis/Freedman*	*arrays in visual perception*
1982	*Dovbysh/Sudakov*	*exchangeable arrays*
1986	*Hestir*	*exchangeable arrays and sheets*
1988–95	*Kallenberg*	*arrays, sheets, measures, functionals*
1992–	*Ivanoff/Weber*	*exchangeable arrays*
1996	*Olshanski/Vershik*	*rotatable arrays*