On the Conditional Prevision of Bruno De Finetti and its Applications

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Abstract

In this paper a critical analysis and an original formalization of the coherent conditional prevision by Bruno de Finetti is presented. Moreover some generalizations, applications and allied algebraic structures are introduced.

1 Introduction

In the fundamental books on "Teoria delle Probabilità" [1], by *Bruno de Finetti,* two different approaches to the conditional probability and prevision are considered.

One of these starts from the axiomatic of the finitely additive conditional probability. It is based on the Boolean operations on events and is considered in all the books or papers on the subjective probability (see e.g. [2], [3]). This theory is formalized in [4]. Further considerations that confirm the validity of the theory of *de Finetti* are in [5].

The second, in "Appendice", of [1], Vol. 2, pp. 718-723, considers the axiomatic of the coherent conditional prevision and is based on the linear algebra. The mathematical tools of this axiomatic are similar to the ones used in multivariate statistics, fuzzy logic, decision theory, join geometry, and other new mathematical theories.

The aims of this paper are: to present a formalization of the coherent conditional prevision, to give a critical analysis of the fundaments of such theory, and to study some applications or generalizations of the results obtained, useful to the cited mathematical theories that use similar mathematical tools.

We consider a particular formalization of the concepts, matured with many meditations, that extends the ones introduced in [10], so some proposed definitions are different from the ones of the mathematical literature.

The conditional prevision is considered also in the book of *Walley* [6]. Like *de Finetti* and unlike *Walley*, we don't assume the *conglomerative condition* for infinite partitions and in general we don't assume the *σ-additivity*. We consider only conditional random numbers and their relations. Then the relations between conditional and unconditional probabilities have not importance, in particular for our theory no difference is between the cases in which the unconditional probability of the conditioning event is positive or null. We assume as basilar concepts the ones we introduce of *extension*, *simple extension*, *regular* and *semi-regular family* of conditional random numbers, that are the tools for our theory on coherent conditional prevision.

2 On the concept of H-conditional random number

In the sequel we denote with $\emptyset$ the impossible event. For every E, E^c is the contrary of E. For any pair of events (E, H), with $H \neq \emptyset$, like de Finetti's, the conditional event E/H is a proposition that assumes the values: **true**, if EH is verified, **false** is E^cH is verified and **undetermined** if H^c holds.

Let H be a non impossible event.

Definition 1 *A* **partition** Π *of H is a non empty family of non impossible events, pairwise disjoint and such that their union is equal to H.*

Definition 2 *Let Π_1 and Π_2 be two partitions of H. The product $\Pi_1 \circ \Pi_2$ is the partition $\{AB : A \in \Pi_1, B \in \Pi_2, AB \neq \emptyset\}$.*

Definition 3 *Let Π be a partition of H. We call "***H-conditional random function***" with domain $D(F) = \Pi$ any function $F : \Pi \to R$. For any $x \in \mathrm{Im}(\Pi)$, $F^{-1}(x)$ is the union of all the $A \in \Pi : F(A) = x$. The partition, of H, $D^*(F) = \{F^{-1}(x), x \in \mathrm{Im}(\Pi)\}$ is called the* **reduced domain** *of F and the H-conditional random function $F^* : F^{-1}(x) \in D^*(F) \to x \in R$ is the* **reduced form** *of F.*

*If Δ is another partition of H we call "***refinement"** *of F with Δ the H-conditional random function $G : \Pi \circ \Delta \to R$, noted $r(F, \Delta)$, such that:*

(RF) $\forall A \in \Pi, B \in \Delta, \quad AB \neq \emptyset \Longrightarrow G(AB) = F(A)$.

We have the following:

Proposition 1 *Let Φ_H be the set of all the H-conditional random functions. The relation ρ such that, $\forall F_1, F_2 \in \Phi_H$, $F_1 \rho F_2$ iff F_1 and F_2 have the same reduced form is an equivalence relation.*

If $D(F_1) = \Pi$, $D(F_2) = \Delta$ then $F_1 \rho F_2 \Longleftrightarrow r(F_1, \Delta) = r(F_2, \Pi)$.

Definition 4 *A* **H-conditional random number** *X is an equivalence class of Φ_H as to ρ, that is an element of the quotient set $Q_H = \Phi_H/\rho$. If $F \in X$, we write $X = [F]$ and F is said to be a* **representant** *of X. The* **domain** *of X is the reduced domain $D^*(F)$ of the elements of X and the* **image** *of X, $\mathrm{Im}(X)$, is the image of the elements of X. We denote with X^* the reduced form of the elements of X, called also* **reduced form** *of X.*

Definition 5 *Let F_1, with domain Π, and F_2 with domain Δ, be two H-conditional random functions. We write $F_1 \leq F_2$ iff $r(F_1, \Delta) \leq r(F_2, \Pi)$.*

Proposition 2 *Let X and Y be two H-conditional random numbers. If there exist $F_1 \in X, F_2 \in Y : F_1 \leq F_2$, then for any $G_1 \in X, G_2 \in Y$ we have $G_1 \leq G_2$, and we write $X \leq Y$. The $\leq$ is an order relation on Q_H.*

Definition 6 *A H-conditional random number X is said to be:*
- **constant** *or* **degenerate** *if $\mathrm{Im}(X)$ is a singleton $\{a\}$, we write $X = a$;*
- **bounded** *or* **limited** *if $\mathrm{Im}(X)$ is bounded;*
- **finite** *if $\mathrm{Im}(X)$ is finite;*
- **H-event** *if $\mathrm{Im}(X) \subseteq \{0, 1\}$;*
- **fuzzy H-event** *if $\mathrm{Im}(X) \subseteq [0, 1]$.*

Usually, if there is not possibility of misinterpretations, a H-event X is identified with the conditional event $X^{-1}(1)/H$.

If X is a bounded H-conditional random number, we put:

$\inf(X) = \inf(\mathrm{Im}(X))$ and $\sup(X) = \sup(\mathrm{Im}(X))$.

Definition 7 *Let σ be an operation on R. The extension of σ to Φ_H, noted also σ, is the function*

$$\sigma : (F_1, F_2) \in \Phi_H \times \Phi_H \to F_1 \sigma F_2,$$

with $F_1 \sigma F_2$ the H-conditional random function such that:

$$\forall A \in D(F_1), B \in D(F_2),\ AB \neq \emptyset,\ (F_1 \sigma F_2)(AB) = F_1(A) \sigma F_2(B).$$

The extension of σ to $R \times \Phi_H$, noted σ^*, is the function

$$\sigma^* : (c, F) \in R \times \Phi_H \to c\sigma^* F,$$

with $c\sigma^* F$ the H-conditional random function such that:

$$\forall A \in D(F), \forall c \in R, (c\sigma^* F)(A) = c\sigma F(A).$$

We have: $\forall F_1, F_2, G_1, G_2 \in \Phi_H$, $(F_1 \rho G_1, F_2 \rho G_2) \implies (F_1 \sigma F_2)\rho(G_1 \sigma G_2)$ and, $\forall c \in R, \forall F, G \in \Phi_H, F\rho G \implies (c\sigma^* F)\rho(c\sigma^* G)$. Then, we can extend σ and σ^* respectively to Q_H and $R \times Q_H$ by putting $\forall F_1, F_2 \in \Phi_H, [F_1]\sigma[F_2] = [F_1 \sigma F_2]$ and $\forall c \in R, \forall F \in \Phi_H, c\sigma^*[F] = [c\sigma^* F]$.

If we consider the usual addition $+$ and multiplication $\cdot$ on R, and we write $*$ for $\cdot^*$, we have:

Proposition 3 *$(Q_H, +, *)$ is a vector space on R. $Q_{H,L}$, set of bounded H-conditional random numbers, and $Q_{H,F}$, set of finite H-conditional random numbers, are subspaces.*

$(Q_H, +, \cdot)$ *is a commutative ring with unity the random H-conditional number equal to* 1, *that is the conditional event H/H. $Q_{H,L}$ and $Q_{H,F}$ are subrings.*

$(Q_H, +, \cdot, *)$ *is an algebra on R. $Q_{H,L}$ and $Q_{H,F}$ are subalgebras.*

3 H-conditional prevision

In the sequel, to avoid undetermined forms, we consider only bounded H-conditional random numbers. Many of results can be extended to the general case.

Let H be a non impossible event.

Definition 8 *Let S_H be a set of bounded H-conditional random numbers. We define* **H-conditional prevision** *on S_H any function $P : S_H \to R$ such that:*

(HCP1) $\forall a, b \in R, \forall X \in S_H, \quad a \le X \le b \Longrightarrow a \le P(X) \le b$;

(HCP2) $\forall X, Y \in S_H, \quad X + Y \in S_H \Longrightarrow P(X + Y) = P(X) + P(Y)$.

*We call "***extension** *of P" any H-conditional prevision on a set S of bounded H-conditional random numbers containing S_H. We say that P is* **coherent** *if there exists an extension of P to the vector space $V(S_H)$ generated by S_H.*

Let S_H be a set of bounded H-conditional random numbers and let P be a H-conditional prevision on S_H. By repeating, with little modifications, for the H-conditional random numbers the proofs of the analogous theorems on the random numbers given in [1] or in [9], we can prove the following:

Proposition 4 *We have:*

(C1) $H/H \in S_H \Longrightarrow P(H/H) = 1$;

(C2) $\forall X \in S_H, \quad X \ge 0 \Longrightarrow P(X) \ge 0$;

(C3) $\forall X, Y \in S_H, \quad X \le Y, Y - X \in S_H \Longrightarrow P(X) \le P(Y)$.

Proposition 5 *If P is coherent then there exists a unique extension P^* of P to $V(S_H)$ and we have:*

(HCP3) $\forall n \in N, \forall X_1, X_2, ..., X_n \in S_H, \forall c_1, c_2, ..., c_n \in R$,

$P^*(c_1X_1 + c_2X_2 + ... + c_nX_n) = c_1P(X_1) + c_2P(X_2) + ... + c_nP(X_n)$.

Proposition 6 *If P is coherent then we have:*

(HCP4) $\forall n \in N, \forall X_1, X_2, ..., X_n \in S_H, \forall c_1, c_2, ..., c_n, c \in R$,

$c_1X_1 + c_2X_2 + ... + c_nX_n \le c \Longrightarrow c_1P(X_1) + c_2P(X_2) + ... + c_nP(X_n) \le c$;

(HCP5) $\forall n \in N, \forall X_1, X_2, ..., X_n \in S_H, \forall c_1, c_2, ..., c_n, c \in R$,

$c_1X_1 + c_2X_2 + ... + c_nX_n \ge c \Longrightarrow c_1P(X_1) + c_2P(X_2) + ... + c_nP(X_n) \ge c$;

(HCP6) $\forall n \in N, \forall X_1, X_2, ..., X_n \in S_H, \forall c_1, c_2, ..., c_n, c \in R$,

$c_1X_1 + c_2X_2 + ... + c_nX_n = c \Longrightarrow c_1P(X_1) + c_2P(X_2) + ... + c_nP(X_n) = c$;

(HCP7) $\forall n \in N, \forall X_1, X_2, ..., X_n \in S_H, \forall c_1, c_2, ..., c_n \in R$,

$\sup[c_1(X_1 - P(X_1)) + c_2(X_2 - P(X_2)) + ... + c_n(X_n - P(X_n))] \ge 0$,

(HCP8) $\forall n \in N, \forall X_1, X_2, ..., X_n \in S_H, \forall c_1, c_2, ..., c_n \in R$,

$\inf[c_1(X_1 - P(X_1)) + c_2(X_2 - P(X_2)) + ... + c_n(X_n - P(X_n))] \le 0$.

Proposition 7 *Let $P : S_H \longrightarrow R$. If at least one among (HCP4), (HCP5), (HCP7) and (HCP8) holds, then P is a coherent H-conditional prevision.*

4 Conditional prevision

4.1 Extensions of H-conditional random numbers

Definition 9 *Let X be a H-conditional random number and let Y be a K-conditional random number, with $H \subseteq K$. We say that Y is a K-***extension** *of X or that X is equal to Y* **conditioned** *to H, we write $X = Y/H$, if:*

(KE1) $D(X) = \{AH \neq \emptyset : A \in D(Y)\}$;

(KE2) $\forall A \in D(Y) : AH \neq \emptyset, X(AH) = Y(A)$.

We denote with $X \uparrow K$ the set of all the extensions of X to K.

We say that Y is a **simple** K**-extension** *of X if:*

(SE1) $Y \in X \uparrow K$;

(SE2) $\exists \overline{A} \in D(Y) : \overline{A} \supseteq K - H$.

If $Y(\overline{A}) = a$ we write $Y = X^{K,a}$ and we say that a is the **value** *of the simple K-extension Y of X.*

We denote with X^K the set of all the simple extensions of X to K. We assume $X = X^{H,a}, \forall a \in R$.

It is easy to prove the:

Proposition 8 *We have:*

(EP1) Y *is a K-conditional random number* $\Longleftrightarrow Y = Y/K$;

(EP2) $Y = Y/K, \emptyset \neq H \subseteq K \Longrightarrow Y/H = YH/H = (Y + aH^c)/H, \forall a \in R$;

(EP3) $\forall a, b \in R, \quad X = X/H, H \subset K \Longrightarrow X^{K,a} = X^{K,b} + (a-b)H^c/K$;

(EP4) $X = X/H \Longrightarrow X \uparrow H = X = X^{H,a}, \forall a \in R$.

Let H and K be two non impossible events with $H \subset K$ and let S_H and S_K sets, respectively, of H-conditional and K-conditional random numbers.

Definition 10 *We say that (S_H, S_K) is* **closed as to the simple K-extensions** *if:*

(CSE1) $H/H \in S_H$, $K/K \in S_K$;

(CSE2) $\forall a \in R, \forall X \in S_H$, $X^{K,a} \in S_K$.

We say that (S_H, S_K) is **semi-closed as to the simple K-extensions** *if it satisfies (CSE1) and*

(CSE2W) $\forall X \in S_H$, $X^{K,0} \in S_K$.

Proposition 9 *If (S_H, S_K) is semi-closed as to the simple K-extensions then $(V(S_H), V(S_K))$ is closed as to the simple K-extensions.*

Proof. If $X \in V(S_H)$, there exist $n \in N$, $a_1, a_2, ..., a_n \in R$, $X_1, X_2, ..., X_n \in S_H$ such that $X = a_1X_1 + a_2X_2 + ... + a_nX_n$. Since (S_H, S_K) is semi-closed as to the simple K-extensions we have $X^{K,0} = a_1X_1^{K,0} + a_2X_2^{K,0} + ... + a_nX_n^{K,0} \in V(S_K)$. Moreover, $H/K = (H/H)^{K,0} \in S_K$ and so $H^c/K = K/K - H/K \in V(S_K)$. Then, from (EP3), $\forall a \in R$, we have $X^{K,a} = X^{K,0} + aH^c/K \in V(S_K)$ ■

Definition 11 *Let P_H and P_K be, respectively, a H-conditional prevision on S_H and a K-conditional prevision on S_K, with $H \subset K$, and such that (S_H, S_K) is closed as to the simple K-extensions. We say that (P_H, P_K) has the property of* **relative coherence** *if:*

(RC) $\quad \forall X \in S_H, \exists a \in R : P_K(X^{K,a}) = P_H(X) = a.$

Remark 1 *By an intuitive point of view, an element X of S_H is interpreted as the winnings of a player* **A** *in a lottery L_H that occurs is H is verified, with stake $P_H(X)$, and $X^{K,a}$ is the winning of* **A** *in a lottery L_K occurring if K is verified, with stake $P_K(X^{K,a})$. Let $G_H = X - P_H(X)$ and $G_K = X^{K,a} - P_K(X^{K,a})$ be the gains of* **A***, respectively with L_H and L_K. If (RC) holds we have $G_H = G_K$ if H is verified and $G_K = 0$ if $K - H$ occurs. If we assume that the gain null is equivalent to not play, then the lotteries L_H and L_K are equivalent.*

In general, if we assume only the condition $P_K(X^{K,a}) = P_H(X)$, we have $G_H = G_K$ if H is verified and there is no lottery if K^c holds. If $K - H$ occurs we have $G_K = a - P_H(X)$ while the lottery L_H is not played. In this case G_K may have many interpretations, e.g. as the **interest** *for the anticipation of the capital $P_H(X)$ if $a > P_H(X)$ or as the* **price** *for the activation of L_K also when $K - H$ occurs if $a < P_H(X)$.*

In this paper we assume L_H and L_K are logically distinct also if (RC) holds.

In the sequel we assume H and K are non impossible events with $H \subset K$, S_H and S_K are two sets, respectively, of H-conditional and K-conditional random numbers, and P_H and P_K are, respectively, a H-conditional prevision on S_H and a K-conditional prevision on S_K.We have:

Proposition 10 *Suppose (S_H, S_K) is closed as to the simple K-extensions and P_H and P_K are coherent. If (P_H, P_K) has the property of relative coherence, P_H^* is the extension of P_H to $V(S_H)$ and P_K^* is the one of P_K to $V(S_K)$, then also (P_H^*, P_K^*) has the property of relative coherence.*

Proof. Let X be an element of $V(S_H)$. Then there exist $n \in N, a_1, a_2, ..., a_n \in R, X_1, X_2, ..., X_n \in S_H : X = a_1X_1 + a_2X_2 + ... + a_nX_n$. Let $I = \{1, 2, ..., n\}$. Since (P_H, P_K) has the property of relative coherence, $\forall i \in I$ there exists $c_i \in R$ such that $P_H(X_i) = P_K(X_i^{K,c_i}) = c_i$. Let $Y = \sum_{i \in I} a_iX_i^{K,c_i}$ and $c = \sum_{i \in I} a_ic_i$. Since P_H and P_K are coherent we have $P_H^*(X) = \sum_{i \in I} a_iP_H(X_i) = \sum_{i \in I} a_iP_K(X_i^{K,c_i}) = P_K^*(Y) = c$ and $Y = X^{K,c} \in V(S_K)$. ■

Definition 12 *Suppose* (S_H, S_K) *is semi-closed as to the simple* K*-extensions. We say that* (P_H, P_K) *has the property of* **multiplicative coherence** *if:*

(MC) $\forall X \in S_H, P_K(X^{K,0}) = P_H(X)P_K(H/K)$.

If $Z \in S_K \cap X \uparrow K$ then $X^{K,0} = ZH/K, X = Z/H$ and so we have:

Corollary 11 *Suppose* (S_H, S_K) *is semi-closed as to the simple* K*-extensions. Then* (P_H, P_K) *has the property of multiplicative coherence if and only if:*

(MCM) $\forall X \in S_H, \forall Z \in S_K \cap X \uparrow K, \quad P_K(ZH/K) = P_H(Z/H)P_K(H/K)$.

Proposition 12 *Suppose that:*

(1) (S_H, S_K) *is semi-closed as to the simple* K*-extensions;*

(2) P_H *and* P_K *are coherent;*

(3) S_K^* *is the union of* S_K *and the set of all the simple* K*-extensions of the elements of* S_H*;*

(4) P_K^* *is an extension of* P_K *to* $V(S_K^*)$.

Then (P_H, P_K^*) *has the property of relative coherence if and only if* (P_H, P_K) *has the property of multiplicative coherence.*

Proof. Assume (P_H, P_K^*) has the property of relative coherence. Then:

$\forall X \in S_H, \exists a \in R : P_K^*(X^{K,a}) = P_H(X) = a$.

From the coherence of P_K^* and (EP3) we have:

$P_K^*(X^{K,a}) = P_K(X^{K,0}) + aP_K^*(H^c/K) = a$, with $a = P_H(X)$, $P_K^*(H^c/K) = 1 - P_K(H/K)$ and so (MC) follows.

On the converse, if (P_H, P_K) has the property of multiplicative coherence, we have (MC). If we put $P_H(X) = a$, we have:

$P_K(X^{K,0}) = a(1 - P_K^*(H^c/K))$.

From (EP3) it is equivalent to $P_K^*(X^{K,a}) = P_H(X) = a$ and so (RC) holds. ∎

4.2 Coherent conditional prevision

We say that X is a **conditional random number** if there exists a non impossible event H such that X is a H-conditional random number. By previous theory we are induced to give the following:

Definition 13 *Let* T *be a non empty family of non impossible events. We say that* $S = \cup_{H \in T} S_H$ *is a* **regular family** *of conditional random numbers if:*

(RF1) $H_1 \in T, H_2 \in T \Longrightarrow H_1 \cup H_2 \in T$;

(RF2) $\forall H \in T$*,* S_H *is a family of* H*-conditional random numbers such that* $H/H \in S_H$*;*

(RF3) $H \in T, K \in T, H \subset K \Longrightarrow (S_H, S_K)$ *is closed as to the simple* K*-extensions.*

We say that S *is a* **semi-regular family** *of conditional random numbers if it satisfies (RF1), (RF2) and:*

(RF3W) $H \in T, K \in T, H \subset K \Longrightarrow (S_H, S_K)$ *is semi-closed as to the simple* K*-extensions.*

Definition 14 *Let* $S = \cup_{H \in T} S_H$ *be a regular family of conditional random numbers. We say that a function* $P : S \longrightarrow R$ *is a* **coherent conditional prevision** *on* S *if:*

(CCP1) $\forall H \in T$, *the restriction* P_H *of* P *to* S_H *is a coherent* H*-conditional prevision;*

(CCP2) $H \in T, K \in T, H \subset K \Longrightarrow (P_H, P_K)$ *has the property of relative coherence.*

Definition 15 *Let* S *be any set of conditional random numbers. We say that a function* $P : S \longrightarrow R$ *is a* **coherent conditional prevision** *on* S *if there exists an extension* P^* *of* P *to a regular family of conditional random numbers* $S^* \supseteq S$ *such that* $P^* : S^* \longrightarrow R$ *is a coherent conditional prevision on* S^*.

By previous definitions and proposition 12 we have:

Proposition 13 *Let* $S = \cup_{H \in T} S_H$ *be a semi-regular family of conditional random numbers. We have that a function* $P : S \longrightarrow R$ *is a coherent conditional prevision on* S *if:*

(CCP1W) $\forall H \in T$, *the restriction* P_H *of* P *to* S_H *is a coherent* H*-conditional prevision;*

(CCP2W) $H \in T, K \in T, H \subset K \Longrightarrow (P_H, P_K)$ *has the property of multiplicative coherence.*

In the sequel, if H and K are events with ? $\neq H \subseteq K$ and X is a H-conditional random number with prevision $P_H(X)$, we put:

$X^{K,*} = X^{K,a}$, with $a = P_H(X)$, for $H \subset K$, $X^{K,*} = X$ for $H = K$.

By definition 13 and propositions 6 and 7 we have the following:

Proposition 14 *Let* T *be a family of non impossible events and let* $S = \cup_{H \in T} S_H$ *be a regular family of conditional random numbers.*

A function $P : S \longrightarrow R$ *is a coherent conditional prevision if and only if:*

(NSC1) $\forall H, K \in T, H \subseteq K \Longrightarrow P_K(X^{K,*}) = P_H(X)$

and at least one of the following properties holds:

(NSC2A) $\forall n \in N, \forall X_i = X_i/H_i \in S, i \in \{1, 2, ..., n\}$, $\forall c_1, c_2, ..., c_n, c \in R$,*if* $K = \cup_{i=1}^n H_i$ *then*

$c_1 X_1^{K,*} + c_2 X_2^{K,*} + ... + c_n X_n^{K,*} \leq c \Longrightarrow c_1 P(X_1) + c_2 P(X_2) + ... + c_n P(X_n) \leq c$;

(NSC2B) $\forall n \in N, \forall X_i = X_i/H_i \in S, i \in \{1, 2, ..., n\}$, $\forall c_1, c_2, ..., c_n, c \in R$,*if* $K = \cup_{i=1}^n H_i$ *then*

$c_1 X_1^{K,*} + c_2 X_2^{K,*} + ... + c_n X_n^{K,*} \geq c \Longrightarrow c_1 P(X_1) + c_2 P(X_2) + ... + c_n P(X_n) \geq c$;

(NSC2C) $\forall n \in N, \forall X_i = X_i/H_i \in S, i \in \{1, 2, ..., n\}$, $\forall c_1, c_2, ..., c_n \in R$,*if* $K = \cup_{i=1}^n H_i$ *then*

$\sup[c_1(X_1^{K,*} - P(X_1)) + c_2(X_2^{K,*} - P(X_2)) + ... + c_n(X_n^{K,*} - P(X_n))] \geq 0$;

(NSC2D) $\forall n \in N, \forall X_i = X_i/H_i \in S, i \in \{1, 2, ..., n\}$, $\forall c_1, c_2, ..., c_n \in R$,*if* $K = \cup_{i=1}^n H_i$ *then*

$\inf[c_1(X_1^{K,*} - P(X_1)) + c_2(X_2^{K,*} - P(X_2)) + ... + c_n(X_n^{K,*} - P(X_n))] \leq 0$.

5 Assessment of coherent conditional previsions

Let $F = \{X_i = X_i/H_i, i \in I = \{1, 2, ..., m\}\}$ be a finite set of finite conditional random numbers and let $K = \cup_{i \in I} H_i$. Denote by D_i^* the set $D(X_i)$ if $H_i = K$ and the set $D(X_i) \cup \{K - H_i\}$ if $H_i \subset K$. The product $D = D_1^* \circ D_2^* \circ ... \circ D_m^*$ is a partition of K and, $\forall i \in I$, any simple K-extension of X_i belongs to $V(D/K)$, vector space generated by the set $\{A/K, A \in D\}$.

Let $D = \{A_j, j \in J\}$ and, $\forall i \in I, j \in J$, $\delta_{ij} = 0$ if $A_j \nsubseteq H_i$ and $\delta_{ij} = 1$ if $A_j \subseteq H_i$. There exist real numbers a_{ij}, $i \in I, j \in J$, such that:

(E1) $X_i^{K,0} = \sum_{j \in J} a_{ij} A_j/K$; (E2) $H_i/K = \sum_{j \in J} \delta_{ij} A_j/K$.

Let T be the set of the finite unions of the elements H_i, $i \in I$. By (E1) and (E2) we have also, $\forall H \in T : H_i \subseteq H$,

(E3) $X_i^{H,0} = \sum_{j \in J} a_{ij} A_j/H$; (E4) $H_i/H = \sum_{j \in J} \delta_{ij} A_j/H$.

For any $H \in T$, we put $S_H = \{X_i^{H,0} : H_i \subseteq H\} \cup \{A_j/H, j \in J\} \cup \{H'/H : H' \in T, H' \subseteq H\} \cup \{\emptyset/H\}$. Since $H \subset H' \Longrightarrow (A_j/H)^{H',0} \in \{\emptyset/H', A_j/H'\}$ we have that $S = \cup_{H \in T} S_H$ is a semi-regular family of conditional random numbers. If P is a coherent conditional prevision on S, then the restriction P_H of P to S_H satisfies the conditions:

(CCP1) $P_H(X_i^{H,0}) = \sum_{j \in J} a_{ij} P_H(A_j/H)$, $\forall i \in I : H_i \subseteq H \in T$;

(CCP2) $\sum_{j \in J} P_H(A_j/H) = 1$, $P_H(A_j/H) \geq 0$, $\forall j \in J$.

The conditions (CCP2) are also sufficient to assure the coherence of the restriction of P_H to the conditional random numbers A_j/H and so the existence of the numbers a_{ij} satisfying (CCP1) with the conditions (CCP2) is necessary and sufficient for the coherence of P_H.

Then, by proposition 13, the coherence conditions on P are:

(CCA) $P(X_i)P(H_i/H) = \sum_{j \in J} a_{ij} P(A_j/H), \forall i \in I : H_i \subseteq H \in T$;

(CCB) $P(H_i/H) = \sum_{j \in J} \delta_{ij} P(A_j/H), \forall H \supseteq H_i, H \in T$;

(CCC) $\sum_{j \in J} P(A_j/H) = 1$, $P(A_j/H) \geq 0, \forall j \in J$.

(CCD) $\forall j \in J, \forall H, H' \in T : H \supseteq H' \supseteq A_j, P(A_j/H')P(H'/H) = P(A_j/H)$.

$\forall\, i \in I$, we put $K_i = \max\{H \in T : H_i \subseteq H, P(H_i/H) > 0\}$. We can prove the:

Proposition 15 *The set $\{K_i, i \in I\}$ is totally ordered as to the inclusion with maximum K and if the conditions (CCA), (CCB), (CCC) and (CCD) hold for $H = K_i$ then, $\forall i \in I$, they are valid also $\forall H : H_i \subseteq H \subseteq K_i$.*

By previous considerations and proposition 15 we deduce an algorithm to verify the coherence of an assessment of values $P(X_i), i \in I$ that extends to the conditional random numbers the one considered in [2] for conditional probabilities.

Suppose we assign the values $P(X_i), i \in I$. If such assessment is coherent, by considering $H = K$ and the $z_j^K = P_K(A_j/K)$ as unknowns, the equations (CCA), (CCB) with the conditions (CCC) must have solutions.

If $Z^K = (z_j^K, j \in J)$ is a solution, for any $i \in I$ we can have two cases:
(a) $P(H_i/K) > 0$, (b) $P(H_i/K) = 0$.

Let $I^* = \{i \in I : P(H_i/K) > 0\}$. By (CCC) $I^* \neq \emptyset$ and, $\forall i \in I^*, H \in T : A_j \subseteq H$, we put $P(A_j/H) = P(A_j/K)/P(H/K)$ with $P(H/K) = \sum_{A_j \subseteq H} P(A_j/K)$. By proposition 15 we have that all the coherence conditions (CCA), (CCB), (CCC) and (CCD) are satisfied for any $i \in I^*$.

For the X_i such that (b) holds, (CCA) and (CCB) are undetermined and we cannot apply proposition 15. We put $F^1 = \{X_i \in F : P(H_i/K) = 0\}$ and $K^1 = \cup\{H_i : X_i \in F^1\}$.

To verify the coherence of the assessment of the $P(X_i), X_i \in F^1$ we must reiterate the previous algorithm by replacing F with F^1 and K with K^1.

By (CCC) it follows that the algorithm, since F is finite, has a finite number of steps and so it permits to conclude if the assessment of the $P(X_i), i \in I$, is coherent or not.

6 New results and applications

6.1 General coherence conditions by a fuzzy partition

Definition 16 *Let $Z_j = Z_j/K$, $j \in J = \{1, 2, ..., n\}$ be fuzzy K-events such that $1 \in \mathrm{Im}(Z_i)$, $\forall j \in J$. We say that the set $\{Z_j, j \in J\}$ is a* **fuzzy partition of K/K** *if:*

(FP1) $\sum_{j\in J} Z_j/K = K/K.$

Proposition 16 *Let $\Pi = \{Z_j = Z_j/K, j \in J = \{1, 2, ..., n\}\}$ be a fuzzy partition of K/K. A function $P : \Pi \longrightarrow R$ is a coherent K-conditional prevision on Π if and only if*

(C1) $\sum_{j\in J} P(Z_j) = 1$, *(C2)* $\forall j \in J, P(Z_j) \geq 0$.

Proof. It is sufficient to prove that, $\forall j \in J, \forall c_j \in R$:

(a) $\inf \sum_{j\in J} c_j[Z_j - P(Z_j)] \leq 0$ or (b) $\sup \sum_{j\in J} c_j[Z_j - P(Z_j)] \geq 0$.

Without loss of generality, we can suppose $c_1 \geq 0$ and $c_1 \geq \max\{|c_j|, j \in J\}$. If we put $Z_1 = 1$, by (C1) and (C2) we have:

$\sum_{j\in J} c_j[Z_j - P(Z_j)] = c_1 - \sum_{j\in J} c_j P(Z_j) \geq 0$

and so (b) holds. ■

Let $F = \{X_i = X_i/H_i, i \in I = \{1, 2, ..., m\}\}$ a finite set of conditional random numbers and let $K = \cup_{i\in I} H_i$. If any $X_i^{K,0}$ and any H_i/K is linearly dependent on the Z_j, then there exist real numbers a_{ij} and b_{ij} such that:

(F1) $X_i^{K,0} = \sum_{j\in J} a_{ij} Z_j$; (F2) $H_i/K = \sum_{j\in J} b_{ij} Z_j$.

If P_K is a coherent K-conditional prevision on the set $\{X_i^{K,0}, H_i/K, i \in I\} \cup \{Z_j, j \in J\}$ then we have:

(FCCP1) $P_K(X_i^{K,0}) = \sum_{j\in J} a_{ij} P_K(Z_j), \forall i \in I,$

(FCCP2) $P_K(H_i/K) = \sum_{j\in J} b_{ij} P_K(Z_j), \forall i \in I,$

with the conditions (C1) and (C2).

If $P : F \to R$ is a real function, we can have the conditions for P is a coherent conditional prevision on F by considerations analogous to the ones of the previous paragraph, by replacing A_j/K with Z_j and δ_{ij} with b_{ij},and with some other suitable modifications.

6.2 Applications to the Decision Theory

In Decision Theory under Uncertainty (see e.g. [7]) a decision maker must choose among a set $A = (A_1, A_2, ..., A_m)$ of acts but the relative desirability of each act depend upon which "*state of nature*", in a set $S = (S_1, S_2, ..., S_n)$, is verified. The desirability is measured by an utility function $u : A \times S \longrightarrow R$ that to any (A_i, S_j) associates a real number u_{ij} that is the *utility* of the act A_i if the state of nature S_j prevails. The states of nature are events that form a partition of the certain event and any act is a random number with domain S. There are two extreme possibilities: the *complete knowledge* of the probabilities *a priori* of the states of nature, in this case we say that we have a decision *under risk*, and the *complete ignorance* of such probabilities. In the first case the best decision criterion is the choice of the act with *maximum utility prevision* and in the second case the choice of the act that *maximizes the minimum payoff*.

A generalization of the classical model is to consider as states of nature the elements of a set $S_j = S_j/K$ of K-conditional random numbers. In this case we have that any act A_i is also a K-conditional random number and a fundamental importance has the choice of a coherent assessment of a K-conditional prevision on the set $A \cup S$.

Particular interesting situation is the one in which S is a fuzzy partition of the certain event, or, in general, of the conditional event K/K.

6.3 Algebraic properties of extensions of conditional random numbers

We assume the fundamental definitions on the hyperstructures given in [8].

Let S be a set of conditional random numbers such that:

(CAE) $\forall X = X/H_1, Y = Y/H_2 \in S$, $(X \uparrow (H_1 \cup H_2)) \cup (Y \uparrow (H_1 \cup H_2)) \subseteq S$.

We can consider the hyperoperation $\circ$ on S such that:

$\forall X = X/H_1, Y = Y/H_2 \in S$, $X \circ Y = (X \uparrow (H_1 \cup H_2)) \cup (Y \uparrow (H_1 \cup H_2))$.

It is easy to prove the:

Proposition 17 $(S, \circ)$ *has the following properties.*

(HA1) $\forall X \in S, X \circ X = X$; *($\circ$ is idempotent)*

(HA2) $\forall X, Y \in S, X \circ Y = Y \circ X$; *(commutativity)*

(HA3) $\forall X, Y, Z \in S, (X \circ Y) \circ Z = X \circ (Y \circ Z)$. *(associativity)*

Then $(S, \circ)$ *is an* **idempotent** *and* **commutative semihypergroup**.

Remark 2 *$(S, \circ)$ is not a hypergroup because there exists $Z \in S : X \circ Z = Y$, with $X = X/H_1$, $Y = Y/H_2$ if and only $H_1 \subseteq H_2$.*

We can consider also the hyperoperation $*$ on S such that:
$\forall X = X/H_1, Y = Y/H_2 \in S$, $X * Y = X^{H_1 \cup H_2} \cup Y^{H_1 \cup H_2}$.
We can prove the:

Proposition 18 *$(S, *)$ has the following properties.*
(HB1) $\forall X \in S, X * X = X$;
(HB2) $\forall X, Y \in S, X * Y = Y * X$;
(HB3) $\forall X, Y, Z \in S, (X * Y) * Z \cap X * (Y * Z) \neq \emptyset$. *(weak associativity)*
Then $(S, *)$ is a **commutative** *and* **weak associative hypergroupoid.**

Remark 3 *Like to the hyperstructure $(S, \circ)$, there exists $Z \in S : X * Z = Y$, with $X = X/H_1$, $Y = Y/H_2$ if and only $H_1 \subseteq H_2$. Unlike $(S, \circ)$, $(S, *)$ has not the associative property because, if $X = X/H_1$ and $H_1 \subset H_2 \subset H_3$ a simple extension to H_3 of a simple extension of X to H_2 is not, in general, a simple extension of X to H_3.*

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