De Finetti's no-Dutch-Book criterion for Gödel logic

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Based on the joint work [1] with Stefano Aguzzoli and Vincenzo Marra

Extended abstract

In the 1930s, Bruno de Finetti described a simple criterion to establish if a distribution of values $\beta(E_i) \in [0,1]$ to events E_i can be extended to a probability measure on the Boolean algebra A generated by such events. The criterion states that for a given β the required extension does *not* exist if and only if it is possible to choose real numbers $\sigma_1, \ldots, \sigma_u, \xi \in \mathbb{R}$ with $\xi > 0$ such that for any truth-value assignment — *i.e.* homomorphism of Boolean algebras — $w\colon A \to \{0,1\}$ one has

$$\sum_{i=1}^{u} \sigma_i(\beta(E_i) - w(E_i)) < -\xi \; . \qquad (1)$$

When this is the case, the assignment $\beta\colon \{E_1, \ldots, E_u\} \to [0,1]$ is called *incoherent*; otherwise, it is *coherent*. See [3], [4].

The intuitive meaning of condition (1) can be stated in terms of a betting metaphor: two players — Ada (the bookmaker) and Blaise (the bettor) — wager money on the possible occurrence of the events in $\mathcal{E} = \{E_1, \ldots, E_u\}$. Ada sets a betting odd $\beta(E_i) \in [0,1]$ for each $E_i \in \mathcal{E}$. Then Blaise chooses a stake $\sigma_i \in \mathbb{R}$. In case $\sigma_i \geq 0$, Blaise hands $\sigma_i\beta(E_i)$ euros to Ada, with the agreement that $\sigma_i w(E_i)$ euros shall be paid back by Ada to Blaise if E_i happens in the possible world w, i.e. if $w(E_i) = 1$. Ada also accepts Blaise's negative stakes $\sigma_i < 0$, to the effect that she must hand $|\sigma_i|\beta(E_i)$ euros to Blaise, with the agreement that $|\sigma_i|w(E_i)$ euros shall be paid back by Blaise to Ada in the possible world w. Hence, the final balance of Ada's *book* $\beta\colon \mathcal{E} \to [0,1]$ is given by

$$\sum_{i=1}^{u} \sigma_i(\beta(E_i) - w(E_i)) \; .$$

Now de Finetti's criterion states that Ada's book is incompatible with the usual Kolmogorov's axioms for a probability measure if and only if it is a *Dutch book*, meaning that it satisfies (1). In words, the book is Dutch if Blaise can choose stakes so that there is a threshold of $\xi > 0$ euros such that in any possible world (*i.e.* whatever the *actual* truth-value of the events E_i is) Ada shall lose more than ξ euros in the final balance. This result underlies de Finetti's theory of probability as (subjective but) rational belief.

It turns out that de Finetti's criterion can also be applied, *mutatis mutandis*, to events described by non-classical logics. See [5] for finite-valued Łukasiewicz logics, [8] for a general result applicable to two-valued but non-Tarskian semantics, [7] for infinite-valued Łukasiewicz logic, and [6] for a class of $[0,1]$-valued logics that includes all logics whose connectives are continuous.

In this abstract we show that de Finetti's criterion can be applied to Gödel propositional logic.

Let $\mathcal{G}_n$ denote the free Gödel algebra over n generators and let $c(n)$ be the cardinality of $\mathcal{G}_n$.

The algebraic counterpart of truth-value assignments to n-variable formulæ of Gödel logic are homomorphisms $w\colon \mathcal{G}_n \to [0,1]$ of Gödel algebras. As in the classical case, it is natural to think of such a function $w \in [0,1]^{c(n)}$ as a *possible world* for Gödel logic. The set of all possible worlds over n variables is written $\mathcal{W}_n$.

A function $\beta\colon \mathcal{G}_n \to [0,1]$ is *incoherent* if there exist (stakes) $\sigma_1, \ldots, \sigma_{c(n)} \in \mathbb{R}$, along with a (threshold) real number $\xi > 0$, such that for any (possible world) $w \in \mathcal{W}_n$ one has (that Ada shall lose more than ξ euros in the final balance)

$$\sum_{i=1}^{c(n)} \sigma_i(\beta(g_i) - w(g_i)) < -\xi\ . \tag{2}$$

Otherwise, β is called *coherent*, or a *de Finetti map* (of n variables). The set of all such de Finetti maps is denoted $\mathcal{D}_n$.

We define a *Kolmogorov map* (of n variables) to be a function $f\colon \mathcal{G}_n \to [0,1]$ satisfying the following axioms.

(C1) $f(\bot) = 0$ and $f(\top) = 1$.

(C2) f preserves order, *i.e.*, $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in \mathcal{G}_n$.

(C3) $f(x \vee y) = f(x) + f(y) - f(x \wedge y)$ for all $x, y \in \mathcal{G}_n$.

We write $\mathcal{K}_n$ for the set of all such Kolmogorov maps. Clearly, each possible world is a Kolmogorov map. Direct inspection shows that the converse fails, except in the trivial case $n = 0$.

If $S \subseteq [0,1]^m$, $m \geq 0$ an integer, we write $\mathrm{conv}S$ for the set of (finite) convex combinations of elements of S. Moreover, we write $\mathrm{cl}S$ for the closure of S in $[0,1]^m$.

We can now state our main result.

Theorem 1 *For any integer $n \geq 0$, we have*

$$\mathcal{K}_n = \mathrm{clconv}\mathcal{W}_n = \mathcal{D}_n\ .$$

In words, a function $f\colon \mathcal{G}_n \to [0,1]$ *satisfies* (C1–C3) *if and only if it is the limit of a sequence of convex combinations of possible worlds if and only if it is a de Finetti map.*

The argument for the proof uses two main ingredients. First, the combinatorial representation of finite Gödel algebras as algebras of parts of a finite forest. Second, the characterisation of $\mathrm{conv}\mathcal{W}_n$ recently obtained in [2] (cfr. Aguzzoli's talk).

References

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