EXCHANGEABILITY AND SEMIGROUPS

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Why semigroups?

They enter the scene naturally, as can be seen already in De Finetti's original result:
Let $P \in M_+^1(\{0,1\}^\infty)$ be exchangeable; then

$$P(x_1,\ldots,x_n) = \varphi_n\left(\sum_{i=1}^n x_i\right) = \varphi\left(\sum_{i=1}^n x_i, n\right) = \varphi\left(\sum_{i=1}^n (x_i, 1)\right)$$

with φ defined on the set

$$S := \left\{(k,n) \in \mathbb{N}_0^2 \mid k \le n\right\}$$

which is (sub-) semigroup inside $\mathbb{N}_0^2$.

Crucial point: φ is a socalled *positive definite function* on S (to be defined below), therefore a (unique) mixture of socalled *characters*, taking here the form

$$(k,n) \stackrel{\sigma}{\longmapsto} p^k q^{n-k}, \quad p,q \in \mathbb{R}$$

where in fact (easy to see) only the characters with $p, q \geq 0$ and $p+q=1$ play a rôle. Inserting this we get

$$\begin{aligned} P(x_1, \ldots, x_n) &= \varphi\left(\sum_{i=1}^{n}(x_i, 1)\right) \\ &= \int \sigma\left(\sum_{i=1}^{n}(x_i, 1)\right) d\mu(\sigma) \\ &= \int \prod_{i=1}^{n} \sigma(x_i, 1) d\mu(\sigma) \\ &= \int_0^1 \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} d\mu(p) \end{aligned}$$

for some (unique) $\mu \in M_+^1([0,1])$, which is De Finetti's result.

Basic definitions and notations

S denotes an abelian semigroup, written additively, with neutral element 0, and possibly with an involution $s \longmapsto s^-$, which in many cases is just the identity.

$\sigma : S \longrightarrow \mathbb{C}$ is a character iff

$$\sigma(s+t) = \sigma(s) \cdot \sigma(t), \sigma(s^-) = \overline{\sigma(s)}, \sigma(0) = 1$$

$\alpha : S \longrightarrow \mathbb{R}_+$ is an absolute value iff

$$\alpha(s+t) \leq \alpha(s) \cdot \alpha(t), \alpha(s^-) = \alpha(s), \alpha(0) = 1$$

$f : S \longrightarrow \mathbb{C}$ is α–bounded iff

$$|f(s)| \leq C \cdot \alpha(s) \quad \forall\, s \in S, \text{ for some } C \geq 0$$

f is exponentially bounded iff it is

$$\alpha\text{–bounded with repect to some absolute value } \alpha$$

$\varphi : S \longrightarrow \mathbb{C}$ is positive definite (abbrev. „p.d.“) iff

$$\sum_{j,k=1}^{n} c_j \overline{c_k} \varphi(s_j + s_k^-) \geq 0 \quad \forall\, n \in \mathbb{N}, c_j \in \mathbb{C}, s_j \in S$$

$\varphi : S \longrightarrow \mathbb{C}$ is completely p.d. („c.p.d.“) iff $s \longmapsto \varphi(s+a)$ is p.d. $\forall\, a \in S$

$S^* :=$ set off all characters of S

$\mathcal{P}(S) :=$ set of all p.d. functions on S

$S^\alpha := \{\sigma \in S^* \mid \sigma \text{ is } \alpha\text{–bounded}\} = \{\sigma \in S^* \mid |\sigma| \leq \alpha\}$

$\mathcal{P}^\alpha(S) := \{\varphi \in \mathcal{P}(S) \mid \varphi \text{ is } \alpha\text{–bounded}\}$

$\hat{S} :=$ all bounded characters on $S = \{\sigma \in S^* \mid |\sigma| \leq 1\}$

$\mathcal{P}^b(S) :=$ all bounded p.d. functions on S

It is easily seen that

$$S^* \subseteq \mathcal{P}_1(S) := \{\varphi \in \mathcal{P}(S) \mid \varphi(0) = 1\}$$

$$S^\alpha \subseteq \mathcal{P}_1^\alpha(S), \quad \varphi \in \mathcal{P}_1^\alpha(S) \Longrightarrow |\varphi| \leq \alpha$$

$$\hat{S} \subseteq \mathcal{P}_1^b(S), \quad \varphi \in \mathcal{P}_1^b(S) \Longrightarrow |\varphi| \leq 1$$

and each $\sigma \in S_+^*$ is even c.p.d.

Theorem of Berg and Maserick

$\mathcal{P}_1^\alpha(S)$ is a Bauer-simplex with S^α as its set of extreme points.

Corollary. If $\varphi \in \mathcal{P}^\alpha(S)$ is c.p.d. then the unique measure representing φ is concentrated on S_+^α.

Let R and S be semigroups, $t : R \longrightarrow S$ with $t(r^-) = (t(r))^-, t(0) = 0,$ and $t(R)$ generating S.

Let $\beta : R \longrightarrow \mathbb{C} \setminus \{0\}$ with $\beta(r^-) = \overline{\beta(r)}$ and $\beta(0) = 1$.

$R^{(\infty)} := \{(r_1, r_2, \ldots) \in R^\infty \mid r_i = 0 \text{ finally}\}$ denotes the direct sum of countably many copies of R.

Let $\varphi : S \longrightarrow \mathbb{C}$ be a given function.

MAIN THEOREM

(i) If $\Phi(r_1, r_2, \ldots) := \prod \beta(r_i) \cdot \varphi\left(\sum t(r_i)\right)$ is p.d. then so is φ .

(ii) If furthermore $|\Phi(r_1, r_2, \ldots)| \leq C \cdot \prod \gamma(r_i)$ for some function $\gamma : R \longrightarrow \mathbb{R}_+, \gamma(0) = 1,$ and some $C > 0$, then

$$\alpha(s) := \inf \left\{ \prod \frac{\gamma(r_i)}{|\beta(r_i)|} \mid \sum t(r_i) = s \right\}$$

is an absolute value on S , φ is α–bounded, and the measure μ representing φ is concentrated on

$$W := \{\sigma \in S^\alpha \mid \beta \cdot (\sigma \circ t) \text{ is p.d. on } R\}$$

(iii) Conversely, for $\mu \in M_+(W)$ and $\varphi(s) := \int \sigma(s) d\mu(\sigma)$ the function Φ as defined in (i) is p.d. and fulfills (ii) for some $C > 0$ and some function γ .

(iv) A corresponding result holds for c.p.d. functions, the measure in (ii) being then concentrated on W_+ .

One of the most direct corollaries is the following result, characterizing spherically exchangeable sequences:

Schoenberg (1938)

$P \in M_+^1(\mathbb{R}^\infty)$ is spherically symmetric

$$\Longleftrightarrow P = \int_0^\infty N(0,c)^\infty d\mu(c) \qquad \exists\, \mu \in M_+^1(\mathbb{R}_+)$$

Proof.

$$\Phi(r_1, r_2, \ldots) := E\left[\exp i\left(\sum r_j X_j\right)\right], (r_1, r_2, \ldots) \in \mathbb{R}^{(\infty)}$$

$$= \varphi\left(\sum r_j^2\right), \text{ for some } \varphi : \mathbb{R}_+ \longrightarrow \mathbb{C}$$

(here $\beta \equiv 1, \gamma \equiv 1, t(r) = r^2$

$$\Longrightarrow \varphi \text{ bounded, p.d., } \varphi(0) = 1 \Longrightarrow \varphi(s) = \int e^{-\lambda s} d\mu(s) \quad \exists\, \mu \in M_+^1([0,\infty])$$

φ continuous $\Longrightarrow \mu(\{\infty\}) = 0$

$$\Longrightarrow \Phi(r_1, r_2, \ldots) = \int_0^\infty e^{-\lambda \sum r_j^2} d\mu(\lambda) \Longrightarrow \text{ result. } \square$$

With only slightly more effort we get the characterization of

Mixtures of the full 2–parameter normal family

$X = (X_1, X_2, \dots)$ real–valued such that
$\Phi(r_1, r_2, \dots) := E\left[\exp i\left(\sum r_j X_j\right)\right] = \varphi\left(\sum r_j, \sum r_j^2\right)$
$\iff P^X = \int_{\mathbb{R}\times\mathbb{R}_+} N(a,c)^\infty d\mu(a,c)$

An example using Laplace- instead of Fourier-transforms is this:
let $X_1, X_2, \dots \geq 0$; then

$$E\left[\exp(-\sum y_j X_j)\right] = \varphi\left(\prod(1+y_j)\right) \text{ , for some } \varphi : [1,\infty[\longrightarrow \mathbb{R}$$
$$\iff P^X = \int_0^\infty \gamma_\lambda^\infty d\mu(\lambda)$$

where $\gamma_\lambda := \frac{1}{\Gamma(\lambda)} x^{\lambda-1} e^{-x} \cdot \lambda_+$, so $\gamma_1 = e_1$ (standard exponential).

And

$$P^X = \int_0^\infty e_\lambda^\infty \, d\mu(\lambda) \qquad [\, e_\lambda \text{ exponential with parameter } \lambda \,]$$

$$\Longleftrightarrow P(X_1 \geq a_1, X_2 \geq a_2, \ldots) = \varphi\left(\sum a_j\right), \text{ some } \varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}$$

De Finetti's theorem in extended form: Let $\mathcal{X}$ be finite or countable, S a semigroup, $t : \mathcal{X} \longrightarrow S$ such that $t(\mathcal{X})$ generates S. $\beta : \mathcal{X} \longrightarrow \,]0, \infty[\,$, $\varphi : S \longrightarrow \mathbb{R}_+$. Then

$P \in M_+^1(\mathcal{X}^\infty)$ fulfills

$$P(x_1, \ldots, x_n) = \prod_1^n \beta(x_i) \cdot \varphi\left(\sum_1^n t(x_i)\right) \qquad \forall\, n, x_i$$

$$\Longleftrightarrow P = \int \kappa_\sigma^\infty \, d\mu(\sigma) \qquad \exists\, \mu \in M_+^1(S_+^*)$$

with μ concentrated on

$$W := \{\sigma \in S_+^* \mid \underbrace{\beta \cdot (\sigma \circ t)}_{=:\kappa_\sigma} \in M_+^1(\mathcal{X})\}$$

EXAMPLES:

1. **The original De Finetti result:**

   $\mathcal{X} = \{0,1\}, S = \{(k,n) \in \mathbb{N}_0^2 \mid k \leq n\}$,
   $t(x) := (x,1), \beta \equiv 1$
   $\sigma(k,n) = p^k q^{n-k}, p, q \geq 0$ is a general non–negative character,
   $\sigma \circ t \in M_+^1(\mathcal{X})$ translates into

$$\sigma(t(0)) + \sigma(t(1)) = \sigma(0,1) + \sigma(1,1) = q + p = 1 \, .$$

2. $\mathcal{X} = \{0,1,2,\ldots,k\}, k \in \mathbb{N}$. Let again $P \in M_+^1(\mathcal{X}^\infty)$ fulfill

$$P(x_1,\ldots,x_n) = \varphi_n\left(\sum_{i=1}^n x_i\right) = \varphi\left(\sum_{i=1}^n (x_i,1)\right)$$

   as before. Then $P = \displaystyle\int_0^1 \kappa_p^\infty d\mu(p)$ with

$$\kappa_p(\{j\}) = p^j q^{k-j}, q = q(p) \text{ from } p^k + p^{k-1}q + \ldots + pq^{k-1} + q^k = 1$$

3. $\mathcal{X} = \mathbb{N}_0$ and P as before. Then $P = \int_0^\infty \gamma_a^\infty d\mu(a)$, γ_a geometric.

4. $\mathcal{X} = \mathbb{N}_0$. We consider $P \in M_+^1(\mathcal{X}^\infty)$ with

$$P(x_1, \ldots, x_n) = \prod_{i=1}^n \frac{1}{x_i!} \cdot \varphi\left(\sum_{i=1}^n (x_i, 1)\right)$$

$$\Longrightarrow P = \int_0^\infty \pi_\lambda^\infty \, d\mu(\lambda), \quad \pi_\lambda \text{ Poisson}$$

Here we have $\beta(x) = 1/x!$. The choice $\beta(x) = 1/(x+1)$ leads to mixtures of

$$\kappa_u(\{x\}) := \frac{u}{-\log(1-u)} \, u^x/(1+x) \; (0 < u < 1) \text{ and } \kappa_0 = \varepsilon_0 \, ,$$

and $\beta(x) := \binom{x+r-1}{r-1}$ leads to negative binomials.

A more abstract result is the

Hewitt-Savage theorem

$\mathcal{X}$ compact, then
$P \in M_+^1(\mathcal{X}^\infty)$ exchangeable $\Longleftrightarrow P = \int \kappa^\infty d\mu(\kappa) \qquad \exists\, \mu \in M_+^1(M_+^1(\mathcal{X}))$

Proof. $\mathcal{F} := \{f : \mathcal{X} \longrightarrow [0,1] \mid f$ is continuous$\}$. Then, with $\delta_f := 1_{\{f\}}$,

$$E\left[\prod f_j(X_j)\right] = \varphi\left(\sum \delta_{f_j}\right)$$

is c.p.d. and bounded, hence so is φ on $\mathbb{N}_0^{(\mathcal{F})}$, the free abelian semigroup over $\mathcal{F}$

$$\Longrightarrow \qquad \varphi\left(\sum \delta_{f_j}\right) = \int \prod \tau(f_j) d\mu(\tau),\ \mu \in M_+^1\left([0,1]^{\mathcal{F}}\right)$$

It is easy to see that μ is concentrated on

$$T := \{\tau : \mathcal{F} \longrightarrow [0,1] \mid \tau(1) = 1, \tau \text{ finitely additive}\}$$

and each $\tau \in T$ extends to a positive linear functional on $C(\mathcal{X})$, i.e. τ can be identified with a Radon probability measure on $\mathcal{X}$. Inserting this above gives the wanted result. $\square$

Remark 1. If $\mathcal{X}$ is just a measurable space then with $\mathcal{F} := \{f : \mathcal{X} \longrightarrow [0,1] \mid f$ is measurable$\}$ one obtains

$$E\left[\prod f_j(X_j)\right] = \int \prod \tau(f_j) d\mu(\tau), \;\; \mu \in M_+^1(T)$$

with

$$T := \{\tau : \mathcal{F} \longrightarrow [0,1] \mid \tau(1) = 1, \tau \text{ additive}\}$$

which is a „weak“ form of a general De Finetti type result.

Remark 2. As noted above, the Berg/Maserick theorem is an essential ingredient in the proof of the main theorem. It can however also be deduced from it:

If $\varphi : S \longrightarrow \mathbb{C}$ is p.d. and α–bounded then $\Phi(s_1, s_2, \ldots) := \varphi\left(\sum s_j\right)$ is p.d., and

$$\mid \Phi(s_1, s_2, \ldots) \mid \leq C \cdot \prod \alpha(s_j) \; .$$

With $R = S, t = id_S$ and $\beta \equiv 1$ the set W in the main theorem reduces to S^α .

Remark 3. The main theorem can be looked at as a result on *exchangeable p.d. functions* (here for simplicity we assume S without involution): if $\Phi : R^{(\infty)} \longrightarrow \mathbb{R}$ is p.d. and exchangeable, then

$$\Phi(r_1, r_2, \ldots) = \varphi\left(\sum \delta_{r_j}\right)$$

with $\varphi : \mathbb{N}_0^{(R)} \longrightarrow \mathbb{R}$ (and $\delta_0 := 0$) . Then φ is p.d., and if $|\Phi(r_1, r_2, \ldots)| \leq C \cdot \prod \gamma(r_j)$, the function φ is α-bounded with $\alpha(\delta_r) := \gamma(r)$. So

$$\varphi\left(\sum \delta_{r_j}\right) = \int \sigma\left(\sum \delta_{r_j}\right) d\mu(\sigma)$$

where μ is a Radon measure on

$$\begin{aligned} W \;&:=\; \{\tau : R \longrightarrow \mathbb{R} \mid r \longmapsto \sigma(\delta_r) =: \tau(r) \text{ p.d. on } R \text{ and } |\tau| \leq \gamma\} \\ &=\; \{\tau \in \mathcal{P}_1(R) \mid |\tau| \leq \gamma\}\ , \end{aligned}$$

leading to

$$\Phi(r_1, r_2, \ldots) = \int_W \prod \tau(r_j) d\mu(\tau)\ ,$$

a mixture of tensor powers of p.d. functions on R .

Let's take another look at the Main Theorem (with $\beta \equiv 1$) :

$$\Phi(r_1, r_2, \ldots) = \varphi\left(\sum t(r_j)\right)$$

with the conclusion Φ p.d. $\Longrightarrow \varphi$ p.d.

Put $U := R^{(\infty)}, \psi(r_1, r_2, \ldots) := \sum t(r_j)$, then $\psi : U \longrightarrow S$ is onto and the theorem says : $\boxed{\varphi \circ \psi \text{ p.d. } \Longrightarrow \varphi \text{ p.d.}}$

What is the crucial property of ψ enabling this conclusion?

> $\forall$ finite subsets $\{s_1, \ldots, s_n\} \subseteq S$ and $\{u_1, \ldots, u_m\} \subseteq U$ and
>
> $\forall\, N \in \mathbb{N} \quad \exists\, \{u_{jp\alpha} \mid j \leq n, p \leq m, \alpha \leq N\} \subseteq U$ such that
>
> $$\psi(u_{jp\alpha} + u^-_{kq\beta}) = s_j + s^-_k + \psi(u_p + u^-_q) \text{ for } (j, p, \alpha) \neq (k, q, \beta)$$

If this is fulfilled, and $\psi(0) = 0$, we call ψ strongly almost additive.
This holds for example if ψ is a homomorphism and onto, but this case is not too interesting.

THEOREM. Let U, S be two semigroups, $\psi : U \longrightarrow S$ be strongly almost additive, and $\varphi : S \longrightarrow \mathbb{C}$ bounded. Then

$$\varphi \circ \psi \text{ p.d. } \Longrightarrow \varphi \text{ p.d.}$$

and φ is in fact a mixture of characters in

$$\hat{S}_\psi := \{\sigma \in \hat{S} \mid \sigma \circ \psi \text{ p.d.}\}$$

(n.b.: a compact subsemigroup of $\hat{S}$).

Furthermore:

$$\{\varphi : S \longrightarrow \mathbb{C} \mid \varphi \text{ bounded}, \varphi(0) = 1, \varphi \circ \psi \text{ p.d.}\}$$

is a Bauer simplex with $\hat{S}_\psi$ as extreme points.

Application to exchangeable random partitions

$V = \{v_1, v_2, \ldots\}$ is a partition of $\mathbb{N} :\Longleftrightarrow v_j \neq \phi, v_j \cap v_k = \phi$ for $j \neq k$, and $\bigcup_j v_j = \mathbb{N}$.
For example $\{\{i\} \mid i \in \mathbb{N}\}$ or $\{\mathbb{N}\}$, the two „extreme“ partitions of $\mathbb{N}$.

$\mathcal{P}$:= set of all partitions of $\mathbb{N}$

$V \in \mathcal{P}$ can be identified with the equivalence relation $E(V) := \bigcup_{v \in V} v \times v \subseteq \mathbb{N}^2$ or with $1_{E(V)} \in \{0,1\}^{\mathbb{N}^2}$, this last identification defining the (natural) topology on $\mathcal{P}$, turning it into a compact metric space .

For $A \subseteq \mathbb{N}$ and $V \in \mathcal{P}$ we write

$$A \sqsubseteq V :\Longleftrightarrow \exists\, v \in V \text{ with } A \subseteq v$$

(that is: A is not separated by the classes of V) .

For $U, V \in \mathcal{P}$ we define

$$U \leq V :\Longleftrightarrow u \sqsubseteq V \quad \forall\, u \in U \quad [\Longleftrightarrow E(U) \subseteq E(V)]$$

Every subset of $\mathcal{P}$ has a unique minimal element w.r.t. „$\leq$ “, and for a family $\mathcal{A}$ of subsets of $\mathbb{N}$ there is a smallest $W \in \mathcal{P}$ such that $A \sqsubseteq W$ for each $A \in \mathcal{A}$. In the particular case of $\mathcal{A} = U \cup V$ for $U, V \in \mathcal{P}$ we write $U \vee V$ for this minimum, and call it (of course) their maximum.

The order intervals $P_U := \{W \in \mathcal{P} \mid U \leq W\}$ fulfill $P_U \cap P_V = P_{U \vee V}$. For $U \in \mathcal{P}$ the classes $u \in U$ with $|u| \geq 2$ are called *non-trivial*, their union $\langle U \rangle$ is called the support of U. Obviously $\langle U \vee V \rangle \subseteq \langle U \rangle \cup \langle V \rangle$, so that

$$\mathcal{U} := \{U \in \mathcal{P} \mid \langle U \rangle \text{ is finite}\}$$

is a subsemigroup w.r.t. „$\vee$ “, with neutral element $U_0 = \{\{j\} \mid j \in \mathbb{N}\}$. The order intervals P_U for $U \in \mathcal{U}$ are clopen and generate the Borel sets of $\mathcal{P}$. Probability measures on $\mathcal{P}$ will be called random partitions.

THEOREM. $\varphi : \mathcal{U} \longrightarrow \mathbb{R}$ is p.d. and normalized (i.e. $\varphi(U_0) = 1$)
$\iff$ $\exists$ (unique) random partition $\mu \in M_+^1(\mathcal{P})$ with

$$\varphi(U) = \mu(P_U) \quad \forall\, U \in \mathcal{U}\,.$$

[easy direction „$\Longleftarrow$“:

$$\sum_{j,k=1}^{n} c_j c_k \varphi(U_j \vee U_k) = \int \left(\sum_{j=1}^{n} c_j 1_{P_{U_j}} \right)^2 d\mu \ \geq 0 \,.]$$

A permutation π of $\mathbb{N}$ induces $\overline{\pi} : \mathcal{P} \longrightarrow \mathcal{P}, \overline{\pi}(V) := \{\pi(v) \mid v \in V\}$, and $\overline{\pi}$ is continuous.

Definition. $\mu \in M_+^1(\mathcal{P})$ is exchangeable: $\iff \mu^{\overline{\pi}} = \mu \quad \forall \, \pi$.

Now $\mu^{\overline{\pi}}(P_U) = \mu(P_{\overline{\pi^{-1}}(U)})$, so μ is exchangeable iff

$$\mu(P_U) = \mu(P_V)$$

$\forall \, U, V \in \mathcal{U}$ with $|\{u \in U \mid |u| = k\}) = |\{v \in V| \, |v| = k\}|$ for $k = 2, 3, \ldots$
iff $\mu(P_U) = \varphi \circ g(U)$ for some φ defined on

$$S := \mathbb{N}_0^{(\{2,3,\ldots\})} \,,$$

with $g(U) := \sum_{\substack{u \in U \\ |u| \geq 2}} \delta_{|u|}$.

This function $g : \mathcal{U} \longrightarrow S$ is in fact strongly almost additive; therefore the

THEOREM (Kingman)

$M_+^{1,e}(\mathcal{P}) := \{\mu \in M_+^1(\mathcal{P}) | \mu \text{ exchangeable}\}$ is a Bauer simplex whose extreme points are precisely those μ for which

$$\mu(P_U) = \sigma(g(U)),\ U \in \mathcal{U} \quad \text{ with } \sigma \in \hat{S}_+ \,.$$

Such a character σ is given by a sequence $(t_2, t_3, \ldots)$ in $[0,1]$.

We see that

$$t_n = \mu(P_{\{\{1,\ldots,n\},\{n+1\},\{n+2\},\ldots\}}), \quad n \geq 2$$

is the μ–probability for $\{1,\ldots,n\}$ not getting separated. For general $U \in \mathcal{U}$ the multiplicativity of σ is reflected in a certain pattern of independence:

$$\mu(P_U) = \prod_{\substack{u \in U \\ |u| \geq 2}} t_{|u|} \,.$$

Kingman showed that there exists $x = (x_1, x_2, \ldots)$ with $x_i \geq 0, \sum x_i \leq 1$, such that

$$t_n = \sum_{i=1}^{\infty} x_i^n \quad \text{for} \quad n = 2, 3, \ldots .$$

There is in fact a natural way to get this distribution μ :

put $x_0 := 1 - \sum_{i=1}^{\infty} x_i$ and let $X_1, X_2, \ldots$ be iid with $P(X_1 = i) = x_i, i \geq 0$.

Then

$$G := \{\{j \in \mathbb{N} \mid X_j = c\} \mid c \in \mathbb{N}\} \cup \{\{i\} \mid X_i = 0\} \smallsetminus \{\emptyset\}$$

is $\mathcal{P}$–valued with distribution μ .

APPROXIMATION LEMMA FOR P.D. MATRICES

$$A_n = \begin{pmatrix} \overbrace{\begin{pmatrix} * & & & & a_{11} \\ & * & & & \\ & & * & & \\ & & & * & \\ a_{11} & & & & * \end{pmatrix}}^{n} & \overbrace{\begin{pmatrix} & & \\ & a_{12} & \\ & & \end{pmatrix}}^{n} & \cdots & \overbrace{\begin{pmatrix} & & \\ & a_{1p} & \\ & & \end{pmatrix}}^{n} \\ \begin{pmatrix} & & \\ & a_{21} & \\ & & \end{pmatrix} & \begin{pmatrix} * & & & & a_{22} \\ & * & & & \\ & & * & & \\ & & & * & \\ a_{22} & & & & * \end{pmatrix} & \cdots & \begin{pmatrix} & & \\ & a_{2p} & \\ & & \end{pmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{pmatrix} & & \\ & a_{p1} & \\ & & \end{pmatrix} & \begin{pmatrix} & & \\ & a_{p2} & \\ & & \end{pmatrix} & \cdots & \begin{pmatrix} * & & & & a_{pp} \\ & * & & & \\ & & * & & \\ & & & * & \\ a_{pp} & & & & * \end{pmatrix} \end{pmatrix}$$

$A = (a_{jk}) \in \mathbb{C}^{p \times p}$ is given,
$\forall \quad n \in \mathbb{N}$
$\exists \; A_n \in \mathbb{C}^{pn \times pn}$ as indicated

If A_n is p.d. $\forall \; n$ and

$$\sup_n \left[\max_{j \le pn} A_n(j,j) \right] < \infty$$

then A is p.d..