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ON NONCOMMUTATIVE DISTRIBUTIONAL SYMMETRIES AND DE FINETTI TYPE THEOREMS ASSOCIATED WITH THEM

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Abstract. We prove general de Finetti theorems for classical, free and boolean independence. Our general de Finetti theorems work for non-easy quantum groups, which generalizes a recent work of Banica, Curran and Speicher. For infinite sequences, we will determine maximal distributional symmetries which means the corresponding de Finetti theorem fails if the sequence satisfies more symmetries other than the maximal one. In addition, we define boolean quantum semigroups in analogue of easy quantum groups by universal conditions on matrix coordinate generators and show some boolean analogue of de Finetti theorems.

1. Introduction

The area of distributional symmetries is one of the richest of modern probability theory. The most obvious problem in this area is to characterize the class of objects of a given type with a specified symmetry property. For example, de Finetti's fundamental theorem states that an infinite sequence of random variables, whose joint distribution is invariant under all finite permutations, is conditionally independent and identically distributed. Later, in [6], rotatability and other continuous symmetries were considered by Freedman. One can see [8] for more details.

Exchangeability and rotatability are classical symmetries associate with permutation groups and orthogonal groups. The quantum analogue of permutation and orthogonal groups were given by Wang in [21, 22]. They are compact quantum groups in the sense of Woronowicz' matrix pseudogroups [24, 25]. In [9], by using symmetries associated with quantum permutation groups, Köstler and Speicher discovered a free analogue of classical de Finetti theorem: an infinite sequence of noncommutative random variables are invariant under quantum permutations is equivalent to the fact that the random variables are identically distributed and free with respect to the conditional expectation onto their tail algebra. A free analogue of Freedman's work on rotatability was given by Curran in [4].

In [3], both classical symmetries and quantum symmetries are studied in the "easiness" formalism. Roughly speaking, those structures are quantum groups associated tensor categories of partitions. For each n, it was shown that there are six easy groups which are denoted by S_n, O_n, B_n, H_n, B'_n, S'_n. We will denote the algebras of continuous functions on these groups by $C_s(n)$, $C_o(n)$, $C_b(n)$, $C_h(n)$, $C_{b'}(n)$, $C_{s'}(n)$, respectively. In the quantum aspect, for each n, together with the work of Weber [23], there are seven easy quantum groups which are denoted by $A_s(n)$, $A_o(n)$, $A_b(n)$, $A_h(n)$, $A_{s'}(n)$, $A_{b'}(n)$, $A_{b\#}(n)$. All these algebras are generated by n^2 matrix coordinates $u_{i,j}$'s which satisfy certain relation R. The relations R for $C_*(n)$ and $A_*(n)$ are suitable such that all these algebras are Hopf algebras in the sense of Woronowicz[24]. The distributional symmetries associated with Woronowicz's are defined via coactions of quantum groups on noncommutative polynomials in the sense of Sołtan [14]. Among these symmetries, in [2], Banica, Curran and Speicher studied de Finetti theorems for $C_s(n)$, $C_o(n)$, $C_b(n)$, $C_h(n)$ and $A_s(n)$, $A_o(n)$, $A_b(n)$, $A_h(n)$. In short, these symmetries can characterize independence relations which are classical or free, and can characterize some special distributions which are symmetric, shifted central limit and centered central limit laws. One goal of this paper is to study de Finetti theorems for all compact quantum groups, for classical and free independence, which are either between $C_s(n)$ and $C_o(n)$ or between $A_s(n)$ and $A_o(n)$.

In [13], Ryll-Nardzewski showed that de Finetti theorem holds under the weaker condition of spreadability. Therefore, for infinite sequences of random variables, different symmetries may characterize a same property. Another goal of this paper is to determine that under what conditions the symmetries characterize a same property for infinite sequences. In our compact quantum group framework, we will show that there is no characterization other than what $C_s(n)$, $C_o(n)$, $C_b(n)$, $C_h(n)$ and $A_s(n)$, $A_o(n)$, $A_b(n)$, $A_h(n)$ can characterize. On the other hand, we will show that these symmetries are maximal which means the corresponding de Finetti theorem fails if the sequence satisfies more symmetries other than a maximal one.

In [17, 18], it was shown that there is a unique non-unital independence, which is called boolean independence, in noncommutative probability. The study of distributional symmetries for boolean independence was started in [11]. We constructed a family of quantum semigroups in analogue of Wang's quantum permutation groups and defined their coactions on joint distributions of sequences. It was shown that the distributional symmetries associated those coactions can be used to characterize boolean independence in a proper framework. In a recent work of Hayase[7], by following the idea of Banica and Speicher, many distributional symmetries related to boolean independence were constructed via the category of interval partitions. By using those distributional symmetries, Haysase find de Finetti theorems for a boolean analogue of easy quantum groups. In this paper, we will defined quantum semigroups, which are related to boolean independence in analogue of easy quantum groups via some universal conditions, $B_s(n)$, $B_o(n)$, $B_b(n)$, $B_h(n)$, $B_{s'}(n)$, $B_{b'}(n)$. Our quantum semigroups are quotient algebras of Hayase's. We do not have maximal distributional symmetries for boolean independence, but we provide a way to check de Finetti theorems for some quantum semigroups other than these universal ones.

Our main result is the following de Finetti theorem:

Theorem 1.1. Let $(\mathcal{A}, \phi)$ be a W^*-probability space and $(x_i)_{i\in\mathbb{N}}$ be a sequence of random variables which generate $\mathcal{A}$

- Classical case:
  Suppose that $\mathcal{A}$ is commutative. Let $\{E(n)\}_{n\in\mathbb{N}}$ be a sequence of orthogonal Hopf algebras such that $C_s(n) \subseteq E(n) \subseteq C_o(n)$ for each $n \in \mathbb{N}$. If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $E(n)$ invariant, then there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that
  1. If $E(n) = C_s(n)$ for all n, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and identically distributed with respect to E.
  2. If $C_s(n) \subseteq E(n) \subseteq C_h(n)$ for all n and there exists a k such that $E(k) \neq C_s(k)$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and have identically symmetric distribution with respect to E.
  3. If $C_s(n) \subseteq E(n) \subseteq C_b(n)$ for all n and there exists a k such that $E(k) \neq C_s(k)$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and have identically shifted-Gaussian distribution with respect to E.
  4. If there exist k_1, k_2 such that $E(k_1) \not\subseteq C_h(k_1)$ and $E(k_2) \not\subseteq C_b(k_2)$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and have centered Gaussian distribution with respect to E.
- Free case:
  Suppose ϕ is faithful. Let $\{E(n)\}_{n\in\mathbb{N}}$ be a sequence of orthogonal Hopf algebras such

that $A_s(n) \subseteq E(n) \subseteq A_o(n)$ for each n. If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $E(n)$ invariant, then there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that

1. If $E(n) = A_s(n)$ for all n, then $(x_i)_{i\in\mathbb{N}}$ are freely independent and identically distributed with respect to E.
2. If $A_s(n) \subseteq E(n) \subseteq A_h(n)$ for all n and there exists a k such that $E(k) \neq A_s(k)$, then $(x_i)_{i\in\mathbb{N}}$ are freely independent and have identically symmetric distribution with respect to E.
3. If $A_s(n) \subseteq E(n) \subseteq A_b(n)$ for all n and there exists a k such that $E(k) \neq A_s(k)$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and have identically shifted-semicircular distribution with respect to E.
4. If there exist k_1, k_2 such that $E(k_1) \nsubseteq A_h(k_1)$ and $E(k_2) \nsubseteq A_b(k_2)$, then $(x_i)_{i\in\mathbb{N}}$ are freely independent and have centered semicircular distribution with respect to E.

- boolean case:
  If ϕ is non-degenerated. Let $\{E(n)\}_{n\in\mathbb{N}}$ be a sequence of orthogonal boolean quantum semigroups such that $B_s(n) \subseteq E(n) \subseteq B_o(n)$ for each n. If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $E(n)$ invariant, then there are a W^*-subalgebra(not necessarily contains the unit of $\mathcal{A}$) $\mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that

1. If $E(n) = B_s(n)$ for all n, then $(x_i)_{i\in\mathbb{N}}$ are boolean independent and identically distributed with respect to E.
2. If $B_s(n) \subseteq E(n) \subseteq B_h(n)$ for all n and there exists a k such that $E(k)$ has a quotient algebra $E'(k)$ that $A_s(k) \subsetneq E'(k) \subseteq A_n(n)$, then $(x_i)_{i\in\mathbb{N}}$ are boolean independent and have identically symmetric distribution with respect to E.
3. If $B_s(n) \subseteq E(n) \subseteq B_b(n)$ for all n and there exists a k such that $E(k)$ has a quotient algebra $E'(k)$ that $A_s(k) \subsetneq E'(k) \subseteq A_b(n)$, then $(x_i)_{i\in\mathbb{N}}$ are boolean independent and have identically shifted-Bernoulli distribution with respect to E.
4. If there exist k_1, k_2 such that $E(k_1)$ and $E(k_2)$ have quotient algebras $E'(k_1) \subseteq A_o(k_1)$ and $E'(k_2) \subseteq A_o(k_2)$ such that $E(k_1) \nsubseteq A_h(k_1)$ and $E'(k_2) \nsubseteq A_b(k_2)$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and have centered Bernoulli distribution with respect to E.

The paper is organized as follows. In section 2, we recall some definitions in noncommutative probability and combinatorial tools. In section 3, we recall orthogonal Hopf algebras and study their properties. In section 4, we define boolean quantum semigroups in analogue of easy quantum groups via certain universal conditions. In section 5, we give the proof of our main theorem and show some applications of the main theorem.

2. Preliminaries and examples

In this section, we recall some necessary definitions and notation in noncommutative probability. For further details, see texts [9, 11, 12, 20].

2.1. **Noncommutative probability.** This part is for noncommutative probability theory and universal independence relations.

Definition 2.1. A noncommutative probability space is a pair $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a unital algebra, and $\phi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\phi 1(_{\mathcal{A}}) = 1$. Elements in $\mathcal{A}$ are called noncommutative random variables. $(\mathcal{A}, \phi)$ is a C^*-probability space if $\mathcal{A}$ is a C^*-algebra and ϕ is a state, i.e. norm one positive linear functional. $(\mathcal{A}, \phi)$ is a W^*-probability space if $\mathcal{A}$ is a W^*-algebra and ϕ is a normal state, i.e. W^*-operator continuous state. The elements of $\mathcal{A}$

are called random variables. Let $x \in \mathcal{A}$ be a random variable, the distribution of x is a linear functional μ_x on $\mathbb{C}[X]$ such that

$$\mu_x(P) = \phi(P(x))$$

for all $P \in \mathbb{C}[X]$, where $\mathbb{C}[X]$ is the set of complex polynomials in one variable.

In this paper, we will be working on W^*-probability spaces $(\mathcal{A}, \phi)$. We require $\mathcal{A}$ to be commutative when we work on classical independence. We require ϕ to be faithful when we work on free independence. l When we work on boolean independence, we require ϕ to be non-degenerated, i.e. the GNS representation associated with ϕ is faithful.

Definition 2.2. Let I be an index set. The algebra of noncommutative polynomials in $|I|$ variables, $\mathbb{C}\langle X_i | i \in I\rangle$, is the linear span of 1 and noncommutative monomials of the form $X_{i_1}^{k_1} X_{i_2}^{k_2} \cdots X_{i_n}^{k_n}$ with $i_1 \neq i_2 \neq \cdots \neq i_n \in I$ and all k_j's are positive integers. For convenience, we use $\mathbb{C}\langle X_i | i \in I\rangle_0$ to denote the set of noncommutative polynomials without a constant term. Let $(x_i)_{i\in I}$ be a family of random variables in a noncommutative probability space $(\mathcal{A}, \phi)$. Their joint distribution is a linear functional $\mu : \mathbb{C}\langle X_i | i \in I\rangle \to \mathbb{C}$ defined by

$$\mu(X_{i_1}^{k_1} X_{i_2}^{k_2} \cdots X_{i_n}^{k_n}) = \phi(x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_n}^{k_n}),$$

and $\mu(1) = 1$.

Remark 2.3. In general, the joint distribution depends on the order of the random variables. For example, let $I = \{1, 2\}$, then μ_{x_1,x_2} may not equal μ_{x_2,x_1}. According to our notation, $\mu_{x_1,x_2}(X_1X_2) = \phi(x_1x_2)$, but $\mu_{x_2,x_1}(X_1X_2) = \phi(x_2x_1)$.

Definition 2.4. Let $(\mathcal{A}, \phi)$ be a noncommutative probability space.

- Suppose that $\mathcal{A}$ is commutative. A family of unital subalgebras $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{A}$ are said to be classical independent if

  $$\phi(a_1a_2 \cdots a_n) = \phi(a_1)\phi(a_2) \cdots \phi(a_n),$$

  whenever $a_k \in \mathcal{A}_{i_k}$, $i, ..., i_n$ are pairwisely different . Let $(x_i)_{i\in I}$ be a family of random variables and $\mathcal{A}_i$'s be the unital subalgebras generated by x_i's, respectively. We say the family of random variables $(x_i)_{i\in I}$ are classical independent if the family of unital subalgebras $(\mathcal{A}_i)_{i\in I}$ are classical independent.
- A family of unital subalgebras $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{A}$ are said to be freely independent if

  $$\phi(a_1 \cdots a_n) = 0,$$

  whenever $a_k \in \mathcal{A}_{i_k}$, $i_1 \neq i_2 \neq \cdots \neq i_n$ and $\phi(a_k) = 0$ for all k. Let $(x_i)_{i\in I}$ be a family of random variables and $\mathcal{A}_i$'s be the unital subalgebras generated by x_i's, respectively. We say the family of random variables $(x_i)_{i\in I}$ are freely independent if the family of unital subalgebras $(\mathcal{A}_i)_{i\in I}$ are freely independent.
- A family of (not necessarily unital) subalgebras $\{\mathcal{A}_i | i \in I\}$ of $\mathcal{A}$ are said to be boolean independent if

  $$\phi(x_1x_2 \cdots x_n) = \phi(x_1)\phi(x_2) \cdots \phi(x_n)$$

  whenever $x_k \in \mathcal{A}_{i_k}$ with $i_1 \neq i_2 \neq \cdots \neq i_n$. A set of random variables $\{x_i \in \mathcal{A} | i \in I\}$ are said to be boolean independent if the family of non-unital subalgebras $\mathcal{A}_i$, which are generated by x_i respectively, are boolean independent.

One refers to [5] for more details of boolean product of random variables.

Definition 2.5. An operator valued probability space $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ consists of an algebra $\mathcal{A}$, a subalgebra $\mathcal{B}$ of $\mathcal{A}$ and a $\mathcal{B}-\mathcal{B}$ bimodule linear map $E : \mathcal{A} \to \mathcal{B}$ i.e.

$$E[b_1 a b_2] = b_1 E[a] b_2, \quad E[b] = b$$

for all $b_1, b_2, b \in \mathcal{B}$ and $a \in \mathcal{A}$. According to the definition in [19], we call E a conditional expectation from $\mathcal{A}$ to $\mathcal{B}$ if E is onto, i.e. $E[\mathcal{A}] = \mathcal{B}$. The elements of $\mathcal{A}$ are called random variables.

Since the framework for boolean independence is a non-unital algebra in general, we will not require our operator valued probability spaces to be unital.

Definition 2.6. Given an algebra $\mathcal{B}$, we denote by $\mathcal{B}\langle X\rangle$ the algebra which is freely generated by $\mathcal{B}$ and the indeterminant X. Let 1_X be the identity of $\mathbb{C}\langle X\rangle$, then $\mathcal{B}\langle X\rangle$ is set of linear combinations of the elements in $\mathcal{B}$ and the noncommutative monomials $b_0 X b_1 X b_2 \cdots b_{n-1} X b_n$ where $b_k \in \mathcal{B} \cup \{\mathbb{C}1_X\}$ and $n \geq 0$. The elements in $\mathcal{B}\langle X\rangle$ are called $\mathcal{B}$-polynomials. In addition, $\mathcal{B}\langle X\rangle_0$ denotes the subalgebra of $\mathcal{B}\langle X\rangle$ which does not contain a constant term i.e. the linear span of the noncommutative monomials $b_0 X b_1 X b_2 \cdots b_{n-1} X b_n$ where $b_k \in \mathcal{B} \cup \{\mathbb{C}1_X\}$ and $n \geq 1$.

Operator-valued independence are defined as follows:

Definition 2.7. Given an operator valued probability space $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ such that $\mathcal{A}$ and $\mathcal{B}$ are unital.

- Suppose that $\mathcal{A}$ is commutative. A family of unital subalgebras $\{\mathcal{A}_i \supset \mathcal{B}\}_{i \in I}$ are said to be conditionally independent with respect to E if

  $$E[a_1 \cdots a_n] = E[a_1]E[a_2] \cdots E[a_n],$$

  whenever $a_k \in \mathcal{A}_{i_k}$ and $i_1, ..., i_n$ are pairwisely different. A family of $(x_i)_{i \in I}$ are said to be conditionally independent over $\mathcal{B}$ if the unital subalgebras $\{\mathcal{A}_i\}_{i \in I}$ which are generated by x_i and B respectively are conditionally independent, or equivalently

  $$E[p_1(x_{i_1}) p_2(x_{i_2}) \cdots p_n(x_{i_n})] = E[p_1(x_{i_1})]E[p_2(x_{i_2})] \cdots E[p_n(x_{i_n})],$$

  whenever $i_1, ..., i_n$ are pairwisely different and $p_1, ..., p_n \in \mathcal{B}\langle X\rangle$.

- A family of unital subalgebras $\{\mathcal{A}_i \supset \mathcal{B}\}_{i \in I}$ are said to be freely independent with respect to E if

  $$E[a_1 \cdots a_n] = 0,$$

  whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, $a_k \in \mathcal{A}_{i_k}$ and $E[a_k] = 0$ for all k. A family of $(x_i)_{i \in I}$ are said to be freely independent over $\mathcal{B}$, if the unital subalgebras $\{\mathcal{A}_i\}_{i \in I}$ which are generated by x_i and B respectively are freely independent, or equivalently

  $$E[p_1(x_{i_1}) p_2(x_{i_2}) \cdots p_n(x_{i_n})] = 0,$$

  whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, $p_1, ..., p_n \in \mathcal{B}\langle X\rangle$ and $E[p_k(x_{i_k})] = 0$ for all k.

- A family of unital subalgebras $\{\mathcal{A}_i \supset \mathcal{B}\}_{i \in I}$ are said to be boolean independent with respect to E if

  $$E[a_1 \cdots a_n] = E[a_1]E[a_2] \cdots E[a_n],$$

  whenever $a_k \in \mathcal{A}_{i_k}$ and $i_1 \neq i_2 \neq \cdots \neq i_n$. A family of random variables $\{x_i\}_{i \in I}$ are said to be boolean independent over $\mathcal{B}$, if the non-unital subalgebras $\{\mathcal{A}_i\}_{i \in I}$ which are generated by x_i and B respectively are boolean independent, or equivalently

  $$E[p_1(x_{i_1}) p_2(x_{i_2}) \cdots p_n(x_{i_n})] = E[p_1(x_{i_1})]E[p_2(x_{i_2})] \cdots E[p_n(x_{i_n})],$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$ and $p_1, ..., p_n \in \mathcal{B}\langle X\rangle_0$.

2.2. **Partitions and cumulants.** All these three independence relations have rich combinatorial theories which we will recall in the follows. One can see [1, 10, 16] for details.

Definition 2.8. Let S be an ordered set:

1. A partition π of a set S is a collection of disjoint, nonempty sets $V_1, ..., V_r$ such that the union of them is S. $V_1, ..., V_r$ are blocks of π. The collection of all partitions o S will be denoted by $P(S)$
2. Given two partitions π, σ, we say $\pi \leq \sigma$ if each block of π is contained in a block of σ.
3. A partition $\pi \in P(S)$ is noncrossing if there is no quadruple (s_1, s_2, r_1, r_2) such that $s_1 < r_1 < s_2 < r_2$, $s_1, s_2 \in V$, $r_1, r_2 \in W$ and V, W are two different blocks of π.
4. A partition $\pi \in P(S)$ is interval if there is no triple (s_1, s_2, r) such that $s_1 < r < s_2$, $s_1, s_2 \in V$, $r \in W$ and V, W are two different blocks of π.
5. Let $\mathbf{i} = (i_1, ..., i_k)$ be a sequence of indices of I and $[k] = \{1, ..., k\}$. We denote by $\ker \mathbf{i}$ the element of $P([k])$ whose blocks are the equivalence classes of the relation
$$s \sim t \Leftrightarrow i_s = i_t$$

Remark 2.9. In this paper, we are interested in $S = \{1, ..., k\}$ for some $k \in \mathbb{N}$. It is easy to see that interval partitions are noncrossing.

Definition 2.10. Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator valued probability space:

1. A $\mathcal{B}$-functional is a n-linear map $\rho : \mathcal{A}^n \to \mathcal{B}$ such that
$$\rho(b_0 a_1 b_1, a_2 b_2, ..., a_n b_n) = b_0 \rho(a_1, b_1 a_2, ..., b_{n-1} a_n) b_n$$
for all $b_0, ..., b_n \in \mathcal{B} \cup \{1_A\}$.
2. For $k \in \mathbb{N}$, let $\rho^{(k)}$ be a $\mathcal{B}$-functional.
3. If $\mathcal{B}$ is commutative. Given $\pi \in P(n)$, we define a $\mathcal{B}$-functional $\rho^{(\pi)} : \mathcal{A}^n \to \mathcal{B}$ by the formula:
$$\rho^{(\pi)}(a_1, ..., a_n) = \prod_{V \in \pi} \rho(V)(a_1, ..., a_n),$$
where if $V = (i_1 < i_2 < \cdots < i_s)$ is a block of π then
$$\rho(V)(a_1, ..., a_n) = \rho^{(s)}(a_{i_1}, ..., a_{i_s}).$$
4. Given $\pi \in NC(n)$, then a $\rho^{(\pi)} : \mathcal{A}^n \to \mathcal{B}$ can be defined recursively as follows:
$$\rho^{(\pi)}(a_1, ..., a_n) = \rho^{(\pi \setminus V)}(a_1, ..., a_l \rho^{(s)}(a_{l+1,...,a_{l+s}}), a_{l+s+1}, ..., a_n)$$
where $V = (l + 1, l + 2, ..., l + s)$ is an interval block of π.

Remark 2.11. If $\mathcal{B}$ is noncommutative, there is no natural way to compute $\rho^{(\pi)}(a_1, ..., a_n)$ for $\pi \notin NC(n)$.

Definition 2.12. Let $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space:

1. If $\mathcal{B}$ is commutative, then the operator-valued classical cumulants $c_E^{(n)} : \mathcal{A}^n \to \mathcal{B}$ are defined by the classical moment-cumulant formula:
$$E[a_1 \cdots a_n] = \sum_{\pi \in P(n)} c_E^{(\pi)}(a_1, ..., a_n),$$
for all $a_1, ..., a_n \in \mathcal{A}$.

2. The operator-valued free cumulants $\kappa_E^{(k)} : \mathcal{A}^n \to \mathcal{B}$ are defined by the free moment-cumulant formula:
$$E[a_1 \cdots a_n] = \sum_{\pi \in NC(n)} \kappa^{(\pi)}(a_1, ..., a_n),$$
for all $a_1, ..., a_n \in \mathcal{A}$.
3. The operator-valued boolean cumulants $b_E^{(k)} : \mathcal{A}^n \to \mathcal{B}$ are defined by the boolean moment-cumulant formula:
$$E[a_1 \cdots a_n] = \sum_{\pi \in I(n)} b_E^{(\pi)}(a_1, ..., a_n),$$
for all $a_1, ..., a_n \in \mathcal{A}$.

Note that all these three types of cumulants can be resolved recursively, e.g.
$$c_E^{(1)}(a_1) = E[a_1]$$
and
$$c_E^{(n)}(a_1, ..., a_n) = E[a_1 \cdots a_n] - \sum_{\pi \in P(n), \pi \neq 1_n} c_E^{(\pi)}(a_1, ..., a_n),$$
where $c_E^{(\pi)}(a_1, ..., a_n)$ depends on $c_E^{(k)}(a_1, ..., a_n)$ for $k = 1, ..., n-1$ if $\pi \neq 1_n$. The same, to determine $\kappa_E^{(n)}$ and $b_E^{(n)}$ we just need to replace $P(n)$ by $NC(n)$ and $I(n)$, respectively.

Theorem 2.13. *Let $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space and $(x_i)_{i \in I}$ be a family of random variables in $\mathcal{A}$:*

1. *If $\mathcal{A}$ is is commutative, then $(x_i)_{i \in I}$ are conditionally independent with respect to E iff*
$$c_E^{(n)}(b_0 x_{i_1} b_1, ..., x_{i_n} b_n) = 0,$$
whenever $i_k \neq i_l$ for some $1 \leq k, l \leq n$.
2. *$(x_i)_{i \in I}$ are free independent with respect to E iff*
$$\kappa_E^{(n)}(b_0 x_{i_1} b_1, ..., x_{i_n} b_n) = 0,$$
whenever $i_k \neq i_l$ for some $1 \leq k, l \leq n$.
3. *$(x_i)_{i \in I}$ are boolean independent with respect to E iff*
$$b_E^{(n)}(b_0 x_{i_1} b_1, ..., x_{i_n} b_n) = 0,$$
whenever $i_k \neq i_l$ for some $1 \leq k, l \leq n$.

Proof. The classical case is well know, the free case is due to Speicher and the scalar boolean case is due to Lehner. For completeness, we provide a sketch of proof to operator-valued boolean case:

If $i_k \neq i_l$ for some $1 \leq k, l \leq n$, then there exists l such that $i_l \neq i_{l+1}$. Therefore, we have
$$\begin{aligned}\sum_{\pi \in I(n)} b_E^{(\pi)}(x_{i_1} b_1, ..., x_{i_n} b_n) &= E[x_{i_1} b_1, ..., x_{i_n} b_n] \\ &= E[x_{i_1} b_1, ..., x_{i_l} b_l] E[x_{i_{l+1}} b_{l+1}, ..., x_{i_n} b_n] \\ &= \sum_{\pi_1 \in I(l)} b_E^{(\pi_1)}(x_{i_1} b_1, ..., x_{i_l} b_l) \sum_{\pi_2 \in I(n-l)} b_E^{(\pi_2)}(x_{i_{l+1}} b_{l+1}, ..., x_{i_n} b_n)\end{aligned}$$
We see that the coefficient of $b_E^{(n}(x_{i_1} b_1, ..., x_{i_n} b_n)$ on the right is 0 which implies that $b_E^{(n}(x_{i_1} b_1, ..., x_{i_n} b_n) = 0$.

□

Definition 2.14. Let $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space. Two random variables $x_1, x_2 \in \mathcal{A}$ are said to be conditionally(free, boolean) i.i.d. respect to E if they are conditionally(free, boolean) independent and have a same distribution. Suppose $x_1, x_2 \in \mathcal{A}$ are conditionally(free boolean) i.i.d. x_1 is said to be symmetric if x_1 and $-x_1$ have a same distribution. x_1 is said to be Gaussian (semicircular, Bernoulli) distributed if x_1 and $\alpha x_1 + \beta x_2$ have a same distribution whenever α, β are real numbers such that $\alpha^2 + \beta^2 = 1$. x_1 is shifted Gaussian (semicircular, Bernoulli) distributed if $x_1 - b$ is Gaussian (semicircular, Bernoulli) distributed for some $b \in \mathcal{B}$.

Remark 2.15. Gaussian (semicircular, Bernoulli) distribution in Definition2.14 is equivalent to the usually definition which is also equivalent to the following cumulants definition. In scalar case for free independence and classical independence, the tail algebra can be considered as the commutative algebra generated by the unit of the probability space. Therefore, the shifted constant commutes with random variables. Graphically, density functions of shifted scalar Gaussian(Semicircular) laws are density functions of centered Gaussian(Semicircular) laws translated by a constant. For example, the density function of the centered semicircular law with variance 1 is

$$\frac{1}{2\pi}\sqrt{4 - x^2}$$

on $[-2, 2]$, where the density function of shifted semicircular law with variance 1 are in the form

$$\frac{1}{2\pi}\sqrt{4 - (x - a)^2}$$

on $[-2 + a, 2 + a]$. But, for boolean independence, the tail algebra does not necessarily contain the unit of the space. Therefore, the shifted constant may not commute with random variables. Graphically, density functions of shifted scalar Bernoulli laws are not simply density functions of centered Bernoulli laws translated by a constant. For example, the density function of the centered semicircular law with variance 1 is

$$1/2\delta_{-1} + 1/2\delta_1,$$

where the density function of shifted Bernoulli law are in the form

$$\frac{a\delta_a + b\delta_{-b}}{a + b}$$

for $a, b > 0$.

Theorem 2.16. Let $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and $(x_i)_{i \in I}$ be a family of random variables in $\mathcal{A}$:

1. If $\mathcal{A}$ is is commutative, then the $\mathcal{B}$-valued joint distribution of $(x_i)_{i \in I}$ has the property corresponding to D in the table below iff for any $\pi \in P(n)$.

$$c_E^{(\pi)}(b_0 x_{i_1} b_1, ..., x_{i_n} b_n) = 0,$$

unless $\pi \in D(n)$ and $\pi \leq ker\mathbf{i}$ where $\mathbf{i} = (i_1, ..., i_n)$.

Partitions D	**Joint distribution**
P: All partitions	Classical independent
P_h: Partitions with even block sizes	Classical independent and symmetric
P_b: Partitions with block size 1 or 2	Classical independent and Gaussian
P_2: Pair partitions	Classical independent and centered Gaussian

2. The $\mathcal{B}$-valued joint distribution of $(x_i)_{i\in I}$ has the property corresponding to D in the table below iff for any $\pi \in P(n)$.
$$\kappa_E^{(\pi)}(b_0 x_{i_1} b_1, ..., x_{i_n} b_n) = 0,$$
unless $\pi \in D(n)$ and $\pi \le ker\mathbf{i}$.

Partitions D	**Joint distribution**
P: Noncrossing partitions	Free independent
P_h: Noncrossing Partitions with even block sizes	Free independent and symmetric
P_b: Noncrossing Partitions with block size 1 or 2	Free independent and semicircular
P_2: Noncrossing Pair partitions	Free independent and centered semicircular

3. The $\mathcal{B}$-valued joint distribution of $(x_i)_{i\in I}$ has the property corresponding to D in the table below iff for any $\pi \in P(n)$.
$$b_E^{(\pi)}(b_0 x_{i_1} b_1, ..., x_{i_n} b_n) = 0,$$
unless $\pi \in D(n)$ and $\pi \le ker\mathbf{i}$.

Partitions D	**Joint distribution**
I: Interval partitions	Boolean independent
I_h: Interval partitions with even block sizes	Boolean independent and symmetric
I_b: Interval partitions with block size 1 or 2	Boolean independent and Bernoulli
I_2: Interval pair partitions	Boolean independent and centered Bernoulli

Proof. These results are well know for free case and classical case. For boolean case, one just need to follow the proof for free case and replace noncrossing partitions by interval partitions. □

3. Noncommutative symmetries

In this section, we will recall distributional symmetries for classic independence are free independence from [3].

Definition 3.1. An orthogonal Hopf algebra is a unital C^*-algebra A generated by n^2 selfadjoint elements $\{u_{i.j}|i,j = 1,...,n\}$, such that the following hold:

1. The inverse of $u = (u_{i,j})_{i,j=1,....n} \in M_n(A)$ is the transpose $u^t = (u_{j,i})_{i,j=1,...n}$, i.e. $\sum\limits_{k=1}^{n} u_{i,k}u_{j,k} = \sum\limits_{k=1}^{n} u_{k,i}u_{k,j} = \delta_{i,j}1_A$.
2. $\Delta(u_{i,j}) = \sum\limits_{k=1}^{n} u_{i,k} \otimes u_{k,j}$ determines a C^*-unital homomorphism $\Delta : A \to A \otimes_{min} A$.
3. $\epsilon(u_{i,j}) = \delta u_{i,j}$ defines a homomorphism $\epsilon : A \to \mathbb{C}$.
4. $S(u_{i,j}) = u_{j,i}$ defines a homomorphism $S : A \to A^{op}$.

This definition adapted from the fundamental work of Woronowicz[24]. Following the notion of Wang's free quantum groups in [21, 22], one can define universal algebras A generated by n^2 noncommutative variables $\{u_{i,j}\}_{i,j=1,...,n}$ which satisfy some relations R. Moreover, for suitable choices of R, we will get Hopf algebras in the sense of Woronowicz[24].

In [3], Banica and Speicher found the following conditions which can be used to construct Hopf orthogonal algebras:

Definition 3.2. *A matrix $u = (u_{i,j})_{i,j=1,\ldots,n} \in M_n(A)$ over a C^*-algebra A is called:*

- *Orthogonal, if all entries of u are selfadjoint, and $uu^t = u^t u = 1_n$,*
- *magic, if it is orthogonal, and its entries are projections.*
- *cubic, if it is orthogonal, and $u_{i,j}u_{i,k} = u_{j,i}u_{k,i} = 0$, for $j \neq k$.*
- *bistochastic, if it is orthogonal, and $\sum\limits_{i=1}^{n} u_{i,j} = \sum\limits_{j=1}^{n} u_{k,i} = 1_A$, for all j,k.*
- *magic',if it is cubic, with the same sum on rows and columns.*
- *bistochastic',if it is orthogonal, with the same sum on rows and columns*

The universal algebras associated with the above conditions are defined as follows:

Definition 3.3. *$A_g(n)$ with $g = o,s,h,b,s,s',b'$ is the universal C^*-algebra generated by the entries of a $n \times n$ matrix which is respectively orthogonal, magic, cubic, bistochastic, magic' and bistochastic'. $C_g(n)$ with $g = o,s,h,b,s,s',b'$ is the universal commutative C^*-algebra generated by the entries of a $n \times n$ matrix which is respectively orthogonal, magic, cubic, bistochastic, magic' and bistochastic'.*

Especially, for each n, $A_s(n)$ and $A_o(n)$ are Wang's quantum permutation group and quantum orthogonal group introduced in [22, 21]. $C_g(n)$ can be considered as the abelianization of $A_g(n)$ for all $g = o,s,h,b,s,s',b'$. It should be mentioned here that there are 7 easy quantum groups in total, see [23].

According to the definitions, we have the following diagram:

$$\begin{array}{ccccc} A_o(n) & \longrightarrow & A_{b'}(n) & \longrightarrow & A_b(n) \\ \downarrow & & \downarrow & & \downarrow \\ A_h(n) & \longrightarrow & A_{s'}(n) & \longrightarrow & A_s(n) \end{array}$$

and

$$\begin{array}{ccccc} C_o(n) & \longrightarrow & C_{b'}(n) & \longrightarrow & C_b(n) \\ \downarrow & & \downarrow & & \downarrow \\ C_h(n) & \longrightarrow & C_{s'}(n) & \longrightarrow & C_s(n) \end{array}$$

and

$$A_g(n) \to C_g(n),$$

for $g = o,s,h,b,s,s',b'$. Here, the arrow means that there exists a morphism of orthogonal Hopf algebras $(A,u) \to (B,v)$ which is a C^*-homomorphism from$A$ to B such that $u_{i,j} \to v_{i,j}$. In other words, $(A,u) \to (B,v)$ implies that B is a quotient C^*-algebra of A. We will use $B \subset A$ for $(A,u) \to (B,v)$.

Proposition 3.4. *Let $E(n)$ be an orthogonal Hopf algebra generated by n^2 selfadjoint elements $\{u_{i,j}\}_{i,j=1,\ldots,n}$, then*

1. *If $E(n) \not\subset A_h(n)$, then there exists a j such that $\sum\limits_{k=1}^{n} u_{k,j}^4 \neq 1_{E(n)}$.*
2. *If $E(n) \not\subset A_b(n)$, then there exists a j such that $\sum\limits_{k=1}^{n} u_{k,j} \neq 1_{E(n)}$.*

Proof. 1. Suppose $\sum\limits_{k=1}^{n} u_{k,i}^4 = 1_{E(n)}$, for all i. Since $\sum\limits_{k=1}^{n} u_{k,i}^2 = 1_{E(n)}$ and $u_{k,i}^4 \leq u_{k,i}^2$, we have

$$u_{k,i}^4 = u_{k,i}^2.$$

$(u_{i,j}^2)_{i,j=1,\dots,n}$ is a matrix of orthogonal projections with sum 1 on rows and columns. Therefore,

$$u_{i,j}^2 u_{i,k}^2 = u_{j,i}^2 u_{k,i}^2 = 0$$

for $j \neq k$. Since $u_{i,j}$ and $u_{i,k}$ are selfadjoint, we have

$$u_{i,j} u_{i,k} = u_{j,i} u_{k,i} = 0$$

which implies that $E(n)$ is a quotient algebra of $A_h(n)$. It is a contradiction.

2. Suppose $\sum_{k=1}^{n} u_{k,i} = 1_{E(n)}$, for all i. Then, for each i, we have

$$\sum_{l=1}^{n} u_{i,l} = \sum_{l=1}^{n}\sum_{k=1}^{n} u_{i,l}u_{k,l} = \sum_{k=1}^{n}\sum_{l=1}^{n} u_{i,l}u_{k,l} = \sum_{k=1}^{n} \delta_{i,k} 1_{E(n)} = 1_{E(n)}.$$

Therefore, $E(n)$ is a quotient algebra of $A_b(n)$ which leads to a contradiction. □

Proposition 3.5. *Let $E(n)$ be an orthogonal Hopf algebra generated by n^2 selfadjoint elements $\{u_{i,j}\}_{i,j=1,\dots,n}$ such that $A_s(n) \subset E(n) \subset A_o(n)$. Then, the following hold:*

1. *If $E(n) \subset A_h(n)$ and $E(n) \subset A_b(n)$, then $E(n) = A_s(n)$.*
2. *If $E(n) \not\subset A_h(n)$ and $E(n) \subset A_b(n)$, then $\exists\, i'$ such that*

$$\sum_{k=1}^{n} u_{k,i'}^m \neq 1,$$

for all $m > 2$.

3. *If $E(n) \not\subset A_b(n)$ and $E(n) \subset A_h(n)$, then $\exists\, i'$ such that*

$$\sum_{k=1}^{n} u_{k,i'}^m \neq 1,$$

for all odd numbers m.

4. *If $E(n) \not\subset A_h(n)$ and $E(n) \not\subset A_h(n)$, then $\exists\, i_1', i_2$ such that*

$$\sum_{k=1}^{n} u_{k,i_1'}^m \neq 1,$$

for all $m > 2$, and

$$\sum_{k=1}^{n} u_{k,i_2'} \neq 1,$$

Proof. It is obvious that $\|u_{i,j}\| \leq 1$ for all $i, j = 1, \dots n$.

1. By assumption, we have

$$\sum_{k=1}^{n} u_{i,k} = 1_{E(n)}$$

and

$$u_{i,j} u_{i,k} = 0$$

for $j \neq k$. Therefore,

$$u_{i,j} = u_{i,j} \sum_{k=1}^{n} u_{i,k} = u_{i,j}^2$$

for all i, j. It implies that $E(n)$ is a quotient algebra of $A_s(n)$, so $E(n) = A_s(n)$.

2. By Proposition 3.5, there exists i' such that

$$\sum_{k=1}^{n} u_{k,i'}^4 \neq 1.$$

Therefore, there exists k' such that

$$u_{k',i'}^4 < u_{k',i'}^2$$

which implies that the spectrum of $u_{k',i'}$ contains a number a such that $-1 < a < 1$. Therefore,

$$u_{k',i'}^m < u_{k',i'}^2$$

for all natural number $m > 2$. Hence, we have

$$\sum_{k=1}^{n} u_{k,i'}^m < 1_{E(n)},$$

for $m > 2$.

3. According to Proposition 3.5, there exists i' such that

$$\sum_{k=1}^{n} u_{k,i'} \neq 1.$$

Therefore, there exists k' such that $u_{k',i'}$ is not an orthogonal projection which implies that

$$u_{k',i'}^{2m+1} < u_{k',i'}^{2m}.$$

Thus, we have

$$\sum_{k=1}^{n} u_{k,i'}^{2m+1} < \sum_{k=1}^{n} u_{k,i'}^{2m} = \sum_{k=1}^{n} u_{k,i'}^{m} = 1_{E(n)},$$

4. Combine Case 2 and 3, the proof is complete. □

Following the proof above, we have

Corollary 3.6. *Let $E(n)$ be an orthogonal Hopf algebra generated by n^2 selfadjoint elements $\{u_{i,j}\}_{i,j=1,\dots,n}$ such that $C_s(n) \subset E(n) \subset C_o(n)$. Then, the following hold:*

1. *If $E(n) \subset A_h(n)$ and $E(n) \subset A_b(n)$, then $E(n) = A_s(n)$.*
2. *If $E(n) \not\subset A_h(n)$ and $E(n) \subset A_b(n)$, then $\exists\, i'$ such that $\sum_{k=1}^{n} u_{k,i'}^m \neq 1$, for all $m > 2$.*
3. *If $E(n) \not\subset A_b(n)$ and $E(n) \subset A_h(n)$, then $\exists\, i'$ such that $\sum_{k=1}^{n} u_{k,i'}^m \neq 1$, for all odd numbers m.*
4. *If $E(n) \not\subset A_h(n)$ and $E(n) \not\subset A_h(n)$, then $\exists\, i_1', i_2$ such that $\sum_{k=1}^{n} u_{k,i_1'}^m \neq 1$, for all $m > 2$, and $\sum_{k=1}^{n} u_{k,i_2'} \neq 1$,*

Now, we turn to define noncommutative distributional symmetries by maps of quantum family of Sołtan[15]:

Definition 3.7. Let (A, Δ) be a quantum group and $\mathcal{V}$ be a unital algebra. By a (right) coaction of the quantum group A on $\mathcal{V}$, we mean a unital homomorphism $\alpha : \mathcal{V} \to \mathcal{V} \otimes \mathcal{A}$ such that

$$(\alpha \otimes id_A)\alpha = (id \otimes \Delta)\alpha.$$

Definition 3.8. Given an orthogonal Hopf algebra $E(n)$ generated by $\{u_{i,j}\}_{i,j=1,...n}$, we have a natural coaction α_n of $E(n)$ on $\mathbb{C}\langle X_1,...,X_n\rangle$ such that

$$\alpha_n : \mathbb{C}\langle X_1,...,X_n\rangle \to \mathbb{C}\langle X_1,...,X_n\rangle \otimes E(n)$$

is an algebraic homomorphism defined via $\alpha_n(X_i) = \sum_{k=1}^n X_k \otimes u_{k,i}$ for all $i = 1,...,n$.

Definition 3.9. Given a probability space $(\mathcal{A}, \phi)$, a sequence of random variables $(x_1,...,x_n)$ of $\mathcal{A}$ and an orthogonal Hopf algebra $E(n)$ generated by $\{u_{i,j}\}_{i,j=1,...n}$. We say that the joint distribution $\mu_{x_1,...,x_n}$ of $x_1,...,x_n$ is $E(n)$ invariant if

$$\mu_{x_1,...,x_n}(p)1_{E(n)} = \mu_{x_1,...,x_n} \otimes id_{E(n)}(\alpha_n(p)),$$

for all $p \in \mathbb{C}\langle X_1,...,X_n\rangle$.

Remark 3.10. Noncommutative distributional symmetries, which are associated with $E(n)$ such that $A_s \subset E(n) \subset A_o(n)(C_s \subset E(n) \subset C_o(n))$, will be used to characterize free(classical) type de Finetti theorems.

Proposition 3.11. Given a probability space $(\mathcal{A}, \phi)$ and a sequence of random variables $(x_1,...,x_n)$ of $\mathcal{A}$. $E(n)$ and $F(n)$ are two orthogonal Hopf algebras such that $E_1(n) \subset E_2(n)$. Then, $(x_1,...,x_n)$ is $E_1(n)$-invariant if $E_2(n)$-invariant.

Proof. Let $\{u_{i,j}^{(l)}\}_{i,j=1,...,n}$ be generators of $E_l(n)$ for $l = 1, 2$. Since $E_1(n) \subset E_2(n)$, there exists a C^*-homomorphism $\Phi : E_2(n) \to E_1(n)$ such that

$$\Phi(u_{i,j}^{(2)}) = u_{i,j}^{(1)}$$

for all i, j. $(x_1,...,x_n)$ is $E_2(n)$-invariant is equivalent to that

$$\mu_{x_1,...,x_n}(X_{\mathbf{i}})1_{E_2(n)} = \sum_{\mathbf{j}\in[n]^k} \mu_{x_1,...,x_n}(X_{\mathbf{j}}) \otimes u_{\mathbf{i},\mathbf{j}}^{(2)},$$

for all monomials $X_{i_1} \cdots X_{i_k} \in \mathbb{C}\langle X_1,...,X_n\rangle$. Apply Φ on both sides of the above equation, we get

$$\mu_{x_1,...,x_n}(X_{\mathbf{i}})1_{E_1(n)} = \sum_{\mathbf{j}\in[n]^k} \mu_{x_1,...,x_n}(X_{\mathbf{j}}) \otimes u_{\mathbf{i},\mathbf{j}}^{(1)},$$

which implies that $(x_1,...,x_n)$ is $E_1(n)$-invariant. □

Given an orthogonal Hopf algebra $E(n)$ generated by $\{u_{i,j}\}_{i,j=1,...,n}$. Then , for $k \in \mathbb{N}$, $E(n)$ can be considered as an orthogonal Hopf algebra $E(n,k)$ generated by $\{v_{i,j}\}_{i,j=1,...,n+k}$ such that

$$v_{i,j} = \begin{cases} u_{i,j} & \text{if } i,j \leq n \\ \delta_{i,j}1_{E(n)} & \text{otherwise} \end{cases}$$

We will call $E(n,k)$ the k-th extension of $E(n)$. To study de Finettil theorems for all orthogonal Hopf algebras $E(n)$, we need to extend $E(n)$-invariance condition on n random variables to infinitely many random variables.

Definition 3.12. Given a probability space $(\mathcal{A}, \phi)$, a sequence of random variables $(x_i)_{i\in\mathbb{N}}$ of $\mathcal{A}$ and an orthogonal Hopf algebra $E(n)$ generated by $\{u_{i,j}\}_{i,j=1,...n}$. We say that the joint distribution μ of $(x_i)_{i\in\mathbb{N}}$ is $E(n)$ invariant if the joint distribution of $(x_1,...,x_{n+k})$ is $E(n,k)$-invariant for all $k \in \mathbb{N}$.

4. Quantum semigroups in analogue of easy quantum groups

Inspired by the previous work in [11], we will define distributional symmetries for boolean independent random variables via quantum semigroups. We briefly recall quantum semigroups' definition here: For any C^*-algebras A and B, the set of morphisms $\mathrm{Mor}(A,B)$ consists of all C^*-algebra homomorphisms acting from A to $M(B)$, where $M(B)$ is the multiplier algebra of B, such that $\phi(A)B$ is dense in B. If A and B are unital C^*-algebras, then all unital C^*-homomorphisms from A to B are in Mor(A,B). In [15],

Definition 4.1. By a quantum semigroup we mean a C^*-algebra $\mathcal{A}$ endowed with an additional structure described by a morphism $\Delta \in Mor(\mathcal{A}, \mathcal{A}\otimes\mathcal{A})$ such that

$$(\Delta\otimes id_{\mathcal{A}})\Delta = (id_{\mathcal{A}}\otimes\Delta)\Delta.$$

The quantum semigroups for boolean independence are unital universal C^*-algebras generated by an orthogonal projection $\mathbf{P}$ and entries of $n\times n$ matrices which satisfying certain relation R related to $\mathbf{P}$:

Definition 4.2. Let $u = (u_{i,j})_{i,j=1,\dots,n} \in M_n(\mathcal{A})$ be an $n\times n$ matrix over a C^*-algebra $\mathcal{A}$ and $\mathbf{P}$ be an orthogonal projection in $\mathcal{A}$, the pair $(u,\mathbf{P})$ is called:

1. $\mathbf{P}$-orthogonal, if all entries of u are selfadjoint, and $uu^t\mathbf{P} = u^tu\mathbf{P} = 1_n\otimes\mathbf{P}$ i.e. $\sum\limits_{k=1}^{n} u_{i,k}u_{j,k}\mathbf{P} = \sum\limits_{k=1}^{n} u_{k,i}u_{k,j}\mathbf{P} = \delta_{i,j}\mathbf{P}$.
2. $\mathbf{P}$-magic, if it is $\mathbf{P}$-orthogonal, and the entries of u are projections.
3. $\mathbf{P}$-cubic, if it is $\mathbf{P}$-orthogonal, and $u_{i,j}u_{i,k}\mathbf{P} = u_{j,i}u_{j,k}\mathbf{P} = 0$, for $j\neq k$.
4. $\mathbf{P}$-bistochastic, if it is $\mathbf{P}$-orthogonal, and $\sum\limits_{j=1}^{n} u_{i,j}\mathbf{P} = \sum\limits_{j=1}^{n} u_{k,i}\mathbf{P} = \mathbf{P}$, for all j,k.
5. $\mathbf{P}$-', if $\sum\limits_{j=1}^{n} u_{i,j}\mathbf{P} = \sum\limits_{j=1}^{n} u_{k,i}\mathbf{P}$, for all j,k.
6. $\mathbf{P}$-magic', if it is $\mathbf{P}$-cubic and $\mathbf{P}$-'
7. $\mathbf{P}$-bistochastic',if it is $\mathbf{P}$-orthogonal and $\mathbf{P}$-'

Unlike the situation in quantum groups, these conditions cannot define universal C^*-algebras since they cannot ensure that $u_{i,j}$'s are bounded. Therefore, we need an additional condition to control the norms of $u'_{i,j}$s. We say $(u_{i,j})_{i,=1,\dots n}$ is norm ≤ 1 if the norm $\|(u_{i,j})_{i,j=1,\dots,n}\|$ of the matrix is ≤ 1

Definition 4.3. $B_g(n)$ with $g = o,s,h,b,s,s',b'$ is the unital universal C^*-algebra generated by the entries of a $n\times n$ norm ≤ 1 matrix $(u_{i,j})_{i,=1,\dots n}$ and an orthogonal projection $\mathbf{P}$ which is respectively $\mathbf{P}$-orthogonal, $\mathbf{P}$-magic, $\mathbf{P}$-cubic, $\mathbf{P}$-bistochastic, $\mathbf{P}$-magic' and $\mathbf{P}$-bistochastic'.

On the C^*-algebra $B_g(n)$ with $g = o,s,h,b,s,s',b'$, we can always define a unital C^*-homomorphism

$$\Delta : B_g(n)\to B_g(n)\otimes B_g(n)$$

by the following formulas:

$$\Delta u_{i,j} = \sum_{k=1}^{n} u_{i,k}\otimes u_{k,j}$$

and

$$\Delta\mathbf{P} = \mathbf{P}\otimes\mathbf{P}, \quad \Delta I = I\otimes I.$$

To show the coproduct is well defined, we need the show that the $(\Delta u_{i,j})_{i,j=1,\dots,n}$ and $\mathbf{P}\otimes\mathbf{P}$ satisfy the universal conditions as $(u_{i,j})_{i,j=1,\dots,n}$ and $\mathbf{P}$ do:
Norm condition: If $\|(u_{i,j})_{i,j=1,\dots n}\|\leq 1$, we have

$$\|(\Delta u_{i,j})_{i,j=1,\dots n}\| = \|(\sum_{k=1}^n u_{i,k}\otimes u_{k,j})_{i,j=1,\dots n}\| = \|(u_{i,j}\otimes 1_n)_{i,j=1,\dots n}(1_n\otimes u_{i,j})_{i,j=1,\dots n}\| \leq \|(u_{i,j})_{i,j=1,\dots n}\|^2\leq 1.$$

P-orthogonal: If $\sum_{k=1}^n u_{i,k}u_{j,k}\mathbf{P} = \sum_{k=1}^n u_{k,i}u_{k,j}\mathbf{P} = \delta_{i,j}\mathbf{P}$, then

$$\begin{aligned}
&\sum_{k=1}^n \Delta u_{i,k}\Delta u_{j,k}\Delta\mathbf{P}\\
=\ &\sum_{k=1}^n (\sum_{l=1}^n u_{i,l}\otimes u_{l,k})(\sum_{m=1}^n u_{j,m}\otimes u_{m,k})(\mathbf{P}\otimes\mathbf{P})\\
=\ &\sum_{k=1}^n\sum_{l=1}^n\sum_{m=1}^n u_{i,l}u_{j,m}\mathbf{P}\otimes u_{l,k}u_{m,k}\mathbf{P}\\
=\ &\sum_{l=1}^n\sum_{m=1}^n u_{i,l}u_{j,m}\mathbf{P}\otimes\delta_{m,l}\mathbf{P}\\
=\ &\sum_{l=1}^n u_{i,l}u_{j,l}\mathbf{P}\otimes\mathbf{P}\\
=\ &\delta_{i,j}\mathbf{P}\otimes\mathbf{P}.
\end{aligned}$$

The same we have $\sum_{k=1}^n \Delta u_{k,i}\Delta u_{k,j}\Delta\mathbf{P} = \delta_{i,j}\mathbf{P}\otimes\mathbf{P}$.
P-cubic: Since $u_{i,j}u_{i,k}\mathbf{P} = u_{j,i}u_{j,k}\mathbf{P} = 0$, for $j\neq k$, we have

$$\begin{aligned}
&\Delta u_{i,j}\Delta u_{i,k}\Delta\mathbf{P}\\
=\ &\sum_{l,m=1}^n u_{i,l}u_{i,m}\mathbf{P}\otimes u_{l,j}u_{m,k}\mathbf{P}\\
=\ &\sum_{l=1}^n u_{i,l}u_{i,l}\mathbf{P}\otimes u_{l,j}u_{l,k}\mathbf{P}\\
=\ &0,
\end{aligned}$$

whenever $j\neq k$. Then same, we have

$$\Delta u_{j,i}\Delta u_{j,k}\Delta\mathbf{P} = 0,$$

whenever $j\neq k$.
P-bistochastic: If $\sum_{j=1}^n u_{i,j}\mathbf{P} = \sum_{j=1}^n u_{j,i}\mathbf{P} = \mathbf{P}$, for all $j=1,\dots,n$.

$$\begin{aligned}
&\sum_{j=1}^n \Delta u_{i,j}\Delta\mathbf{P}\\
=\ &\sum_{j=1}^n\sum_{k=1}^n u_{i,k}\mathbf{P}\otimes u_{k,j}\mathbf{P}\\
=\ &\sum_{j=1}^n u_{i,j}\mathbf{P}\otimes\mathbf{P}\\
=\ &\mathbf{P}\otimes\mathbf{P}.
\end{aligned}$$

The same we will have $\sum_{j=1}^{n} \Delta u_{j,i} \Delta \mathbf{P} = \mathbf{P} \otimes \mathbf{P}$, for all j.

P' -condition: Let $r = \sum_{j=1}^{n} u_{i,j}\mathbf{P} = \sum_{j=1}^{n} u_{j,i}\mathbf{P}$, for $j \neq k$.

$$\begin{aligned} & \sum_{j=1}^{n} \Delta u_{i,j} \Delta \mathbf{P} \\ = & \sum_{j,l=1}^{n} u_{i,l}\mathbf{P} \otimes u_{l,j}\mathbf{P} \\ = & \sum_{l=1}^{n} u_{i,l}\mathbf{P} \otimes r \\ = & \; r \otimes r, \end{aligned}$$

for all j. The same we will have $\sum_{j=1}^{n} \Delta u_{j,i} \Delta \mathbf{P} = r \otimes r$ for all j.

Therefore, Δ is a well defined C^*-homomorphism and $(B_g(n), \Delta)$ with $g = o, s, h, b, s, s', b'$ are quantum semigroups. As the relation for easy quantum groups, we have the following diagram for boolean quantum semigroups:

$$\begin{array}{ccccc} B_o(n) & \longrightarrow & B_{b'}(n) & \longrightarrow & B_b(n) \\ \downarrow & & \downarrow & & \downarrow \\ B_h(n) & \longrightarrow & B_{s'}(n) & \longrightarrow & B_s(n) \end{array}$$

We can see that easy quantum groups could be quotient algebras of these easy quantum semigroups with requirement of $\mathbf{P} = 1$. The algebras $\mathcal{B}_g(n)$ generated by the generators of $B_g(n)$ with $g = o, s, h, b$ are quotient algebras of Hayase's Hopf algebras $C(G_n^{I_2}), C(G_n^{I}), C(G_n^{I_h}), C(G_n^{I_b})$ in [7], respectively. Actually, $B_g(n)$ with $g = o, s, h, b$ satisfy Hayase's universal conditions for $C(G_n^{I_2}), C(G_n^{I}), C(G_n^{I_h}), C(G_n^{I_b})$. To check the some vanishing conditions, we need the following notation for convenience: Given $\pi_1 \in I(k_1)$ and $\pi_2 \in I(k_2)$, $\pi = \pi_1\pi_2 \in I(k_1 + k_2)$ denotes the concatenation of π_1 and π_2. Given $\mathbf{j}_1 = (j_1, ..., j_{k_1}) \in [n]^{k_1}$ and $\mathbf{j}_2 = (j'_1, ..., j'_{k_2}) \in [n]^{k_2}$, $\mathbf{j} = \mathbf{j}_1\mathbf{j}_2 = (j_1, ..., j_{k_1}, j'_1, ..., j'_{k_2}) \in [n]^{k_1+k_2}$.

According to Definition 2.8, it is obvious that

Lemma 4.4. Let $\pi \in I(k_1 + k_2)$ such that $\pi = \pi_1\pi_2$ for some $\pi_1 \in I(k_1)$ and $\pi_2 \in P(k_2)$. Let $\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2$ such that $\mathbf{j}_1 \in [n]^{k_1}$ and $\mathbf{j}_2 \in [n]^{k_2}$. Then, $\pi \leq ker\mathbf{j}$ iff $\pi_i \leq \ker \mathbf{j}_i$ for $i = 1, 2$.

Therefore ,we have the following:

Lemma 4.5. Given $\pi_1 \in I(k_1)$, $\pi_2 \in P(k_2)$ and $\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2$ such that $\mathbf{j}_1 \in [n]^{k_1}$ and $\mathbf{j}_2 \in [n]^{k_2}$. If

$$\sum_{\mathbf{i}_i \in [n^{k_i}], \pi_i \leq \ker \mathbf{i}_i} u_{\mathbf{i}_i, \mathbf{j}_i}\mathbf{P} = \begin{cases} \mathbf{P} & \text{if } \pi \leq \ker \mathbf{j}_i \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2$. Then, we have

$$\sum_{\mathbf{i} \in [n^{k_1+k_2}], \pi_1\pi_2 \leq \ker \mathbf{i}} u_{\mathbf{i}, \mathbf{j}}\mathbf{P} = \begin{cases} \mathbf{P} & \text{if } \pi \leq \ker \mathbf{j} \\ 0 & \text{otherwise} \end{cases}$$

Proof. By a direct computation, we have:

$$\sum_{\substack{\mathbf{i}\in[n^{k_1+k_2}]\\ \pi_1\pi_2\le\ker\mathbf{i}}} u_{\mathbf{i},\mathbf{j}}\mathbf{P}=\sum_{\substack{\mathbf{i}_1\in[n^{k_1}]\\ \pi_1\le\ker\mathbf{i}_1}}\sum_{\substack{\mathbf{i}_2\in[n^{k_2}]\\ \pi_2\le\ker\mathbf{i}_2}} u_{\mathbf{i}_1,\mathbf{j}_1}u_{\mathbf{i}_2,\mathbf{j}_2}\mathbf{P}=\begin{cases}\sum\limits_{\substack{\mathbf{i}_1\in[n^{k_1}]\\ \pi_1\le\ker\mathbf{i}_1}} u_{\mathbf{i}_1,\mathbf{j}_1}\mathbf{P} & \text{if } \pi_1\le\ker\mathbf{j}_2\\ 0 & \text{otherwise}\end{cases}$$

Therefore,

$$\sum_{\substack{\mathbf{i}\in[n^{k_1+k_2}]\\ \pi_1\pi_2\le\ker\mathbf{i}}} u_{\mathbf{i},\mathbf{j}}\mathbf{P}\begin{cases}=\mathbf{P} & \pi_1\le\ker\mathbf{j}_1 \text{ and } \pi_2\le\ker\mathbf{j}_2\\ 0 & \text{otherwise}\end{cases},$$

which completes the proof. □

Now, we can turn to check a vanishing condition:

Lemma 4.6. Let $u_{i,j}$'s and $\mathbf{P}$ be the standard generators of $B_o(n)$,$B_s(n)$,$B_h(n)$,$B_b(n)$. Then, we have

$$\sum_{\mathbf{i}\in[n^k],\pi\le\ker\mathbf{i}} u_{\mathbf{i},\mathbf{j}}\mathbf{P}=\begin{cases}\mathbf{P} & \text{if } \pi\le\ker\mathbf{j}\\ 0 & \text{otherwise}\end{cases}$$

for $\pi\in I_2(k), I(k), I_h(k), I_b(k)$, respectively.

Proof. 1. For $B_o(n)$, $k=2$. The identity holds by the definition of $B_o(n)$. Since all partitions in $I_2(n)$ are concatenations of pair partitions by Lemma 4.5, the identity is true.

2. For $B_b(n)$, the identity holds by the definition of $B_b(n)$ when π is a single partition or a partition. Since all partitions in $I_b(n)$ are concatenations of single partitions and pair partitions, by Lemma 4.5, the identity is true.

3. For $B_h(n)$ we just need to check $\pi=1_{2m}\in I_h(2m)$ for all $m\in\mathbb{N}$. It follows that

$$\sum_{\substack{\mathbf{i}\in[n]^{2m}\\ \pi\le\ker\mathbf{i}}} u_{\mathbf{i},\mathbf{j}}\mathbf{P}=\sum_{i=1}^n u_{i,j_1}\cdots u_{i,j_{2m}}\mathbf{P}.$$

It equals zero if $j_l\neq j_{l+1}$ for some l, otherwise

$$\sum_{i=1}^n u_{i,j_1}\cdots u_{i,j_{2m}}\mathbf{P}=\sum_{i=1}^n u_{i,j_1}^{2m}\mathbf{P}=\sum_{i=1}^n u_{i,j_1}^{2m-2}\sum_{l=1}^n u_{l,j_1}^2\mathbf{P}=\sum_{i=1}^n u_{i,j_1}^{2m-2}\mathbf{P}=\cdots=\sum_{l=1}^n u_{l,j_1}^2\mathbf{P}=\mathbf{P}.$$

Since all partitions in $I_b(n)$ are concatenations of blocks of even length, by Lemma 4.5, the identity is true.

4. For $B_s(n)$ we just need to check $\pi=1_m\in I(m)$, for all $m\in\mathbb{N}$. It follows that

$$\sum_{\substack{\mathbf{i}\in[n]^{m}\\ \pi\le\ker\mathbf{i}}} u_{\mathbf{i},\mathbf{j}}\mathbf{P}=\sum_{i=1}^n u_{i,j_1}\cdots u_{i,j_m}\mathbf{P}.$$

It equals zero if $j_l\neq j_{l+1}$ for some l, otherwise

$$\sum_{i=1}^n u_{i,j_1}\cdots u_{i,j_m}\mathbf{P}=\sum_{i=1}^n u_{i,j_1}^{m}\mathbf{P}=\sum_{i=1}^n u_{i,j_1}^{m-1}\sum_{l=1}^n u_{l,j_1}\mathbf{P}=\sum_{i=1}^n u_{i,j_1}^{m-1}\mathbf{P}=\cdots=\sum_{l=1}^n u_{l,j_1}\mathbf{P}=\mathbf{P}.$$

Since all partitions in $I_b(n)$ are concatenations of blocks of arbitrary length, by Lemma 4.5, the identity is true. □

Now, we define noncommutative distributional symmetries for boolean independence in general:

Definition 4.7. An orthogonal boolean quantum semigroup is a unital C^*-algebra A generated by n^2 selfadjoint elements $\{u_{i.j}|i,j=1,...,n\}$ and an orthogonal projection $\mathbf{P}$, such that the following hold:

1. $u=(u_{i,j})_{i,j=1,...,n}\in M_n(A)$ is norm≤ 1 and (u,$\mathbf{P}$) is $\mathbf{P}$-orthogonal.
2. $\Delta(u_{i,j})=\sum\limits_{k=1}^{n}u_{i,k}\otimes u_{k,j}$ and $\Delta\mathbf{P}=\mathbf{P}\otimes\mathbf{P},\Delta I=I\otimes I$ determine a C^*-unital homomorphism $\Delta:A\to A\otimes_{min}A$.

Definition 4.8. Let (A,Δ) be a quantum semigroup and $\mathcal{V}$ be a unital algebra. By a right coaction of the quantum semigroup A on $\mathcal{V}$, we mean a unital homomorphism $\alpha:\mathcal{V}\to\mathcal{V}\otimes\mathcal{A}$ such that

$$(\alpha\otimes id_A)\alpha=(id\otimes\Delta)\alpha.$$

Definition 4.9. Given an orthogonal boolean quantum semigroup $E(n)$ generated by $\{u_{i,j}\}_{i,j=1,...n}$ and $\mathbf{P}$, we have a natural coaction α_n of $E(n)$ on $\mathbb{C}\langle X_1,...,X_n\rangle$ such that

$$\alpha_n:\mathbb{C}\langle X_1,...,X_n\rangle\to\mathbb{C}\langle X_1,...,X_n\rangle\otimes E(n)$$

is an algebraic homomorphism defined via $\alpha_n(X_i)=\sum_{k=1}^n X_k\otimes u_{k,i}$ for all i.

Definition 4.10. Given a probability space $(\mathcal{A},\phi)$, a sequence of random variables $(x_1,...,x_n)$ of $\mathcal{A}$ and an orthogonal boolean quantum semigroup $E(n)$ generated by $\{u_{i,j}\}_{i,j=1,...n}$ and $\mathbf{P}$. We say that the joint distribution $\mu_{x_1,...,x_n}$ of $x_1,...,x_n$ is $E(n)$ invariant if

$$\mu_{x_1,...,x_n}(p)\mathbf{P}=\mu_{x_1,...,x_n}\otimes id_{E(n)}(\alpha_n(p))\mathbf{P},$$

for all $p\in\mathbb{C}\langle X_1,...,X_n\rangle$.

The same as matrix quantum groups, we can define $E(n)$ invariance condition for infinite sequences. Given an orthogonal boolean quantum semigroup $E(n)$ generated by $\{u_{i,j}\}_{i,j=1,...,n}$ and $\mathbf{P}$ then , for $k\in\mathbb{N}$, $E(n)$ can be considered as an orthogonal boolean quantum semigroup $E(n,k)$ generated by $\{v_{i,j}\}_{i,j=1,...,n+k}$ and $\mathbf{P'}$ such that

$$v_{i,j}=\begin{cases} u_{i,j} & \text{if } i,j\leq n\\ \delta_{i,j}1_{E(n)} & \text{otherwise}\end{cases}$$

and $\mathbf{P'}=\mathbf{P}$. We will call $E(n,k)$ the k-th extension of $E(n)$.

Definition 4.11. *Given a probability space $(\mathcal{A},\phi)$, a sequence of random variables $(x_i)_{i\in\mathbb{N}}\in\mathcal{A}$ and an orthogonal Hopf algebra $E(n)$ generated by $\{u_{i,j}\}_{i,j=1,...n}$. We say that the joint distribution μ of $(x_i)_{i\in\mathbb{N}}$ is $E(n)$ invariant if the joint distribution of $(x_1,...,x_{n+k})$ is $E(n,k)$-invariant for all $k\in\mathbb{N}$.*

Proposition 4.12. *Let $(\mathcal{A},\mathcal{B},E:\mathcal{A}\to\mathcal{B})$ be an operator valued probability space and $\{x_i\}_{i=1,...,n}$ be a sequence of random variables in $\mathcal{A}$. Let ϕ be a linear functional on $\mathcal{A}$ such that $\phi(\cdot)=\phi(E[\cdot])$. Then, in probability space $(\mathcal{A},\phi)$, we have*

- *If $\{x_i\}_{i=1,...,n}$ is identically distributed and boolean independent with respect to E, then the sequence is B_s-invariant.*
- *If $\{x_i\}_{i=1,...,n}$ is identically symmetric distributed and boolean independent with respect to E, then the sequence is B_h-invariant.*

- *If $\{x_i\}_{i=1,\ldots,n}$ has identically shifted Bernoulli distribution and is boolean independent with respect to E, then the sequence is B_b-invariant.*
- *If $\{x_i\}_{i=1,\ldots,n}$ has identically centered Bernoulli distribution and boolean independent with respect to E, then the sequence is B_o-invariant.*

Proof. Suppose that the joint distribution of $\{x_i\}_{i=1,\ldots,n}$ satisfies one of the conditions specified in the statement of the proposition, and let $D(k)$ be the partition family associated to the corresponding quantum semigroups. Let $X_{\mathbf{j}} = X_{j_1} \cdots X_{j_k}$, by Lemma 4.6 and 2.16, we have

$$\begin{aligned}
\mu_{x_1,\ldots,x_n}(\alpha_n(X_{\mathbf{j}}))\mathbf{P} &= \sum_{\mathbf{i}\in[n]^k} \mu_{x_1,\ldots,x_n}(X_{\mathbf{i}})u_{\mathbf{i},\mathbf{j}}\mathbf{P} \\
&= \sum_{\mathbf{i}\in[n]^k} \phi(x_{\mathbf{i}})u_{\mathbf{i},\mathbf{j}}\mathbf{P} \\
&= \sum_{\mathbf{i}\in[n]^k} \phi(E[x_{\mathbf{i}}])u_{\mathbf{i},\mathbf{j}}\mathbf{P} \\
&= \sum_{\mathbf{i}\in[n]^k} \sum_{\pi\in D(k)} \phi(b_E^{(\pi)}(x_{\mathbf{i}}))u_{\mathbf{i},\mathbf{j}}\mathbf{P} \\
&= \sum_{\pi\in D(k)} \sum_{\mathbf{i}\in[n]^k} \phi(b_E^{(\pi)}(x_{\mathbf{i}}))u_{\mathbf{i},\mathbf{j}}\mathbf{P} \\
&= \sum_{\pi\in D(k)} \sum_{\substack{\mathbf{i}\in[n]^k\\ \pi\leq\ker\mathbf{i}}} \phi(b_E^{(\pi)}(x_{\mathbf{i}}))u_{\mathbf{i},\mathbf{j}}\mathbf{P} \\
&= \sum_{\pi\in D(k)} \sum_{\substack{\mathbf{i}\in[n]^k\\ \pi\leq\ker\mathbf{i}}} \phi(b_E^{(\pi)}(x_1,\ldots,x_1))u_{\mathbf{i},\mathbf{j}}\mathbf{P} \\
&= \sum_{\substack{\pi\in D(k)\\ \pi\leq\ker\mathbf{j}}} \phi(b_E^{(\pi)}(x_1,\ldots,x_1))\mathbf{P} \\
&= \sum_{\substack{\pi\in D(k)\\ \pi\leq\ker\mathbf{j}}} \phi(b_E^{(\pi)}(x_{\mathbf{j}}))\mathbf{P} \\
&= \phi(E[x_{\mathbf{j}}])\mathbf{P} \\
&= \phi(x_{\mathbf{j}})p \\
&= \mu_{x_1,\ldots,x_n}(X_{\mathbf{j}})\mathbf{P},
\end{aligned}$$

which completes the proof. □

5. Main result

In this section, we will prove Theorem 1. Then, we will present an application of our main theorem to easy quantum groups $C_{s'}(n)$, $C_{b'}(n)$, $A_{s'}(n)$, $A_{b'}(n)$, $A_{b\#}(n)$ and $B_{s'}(n)$, $B_{b'}(n)$.

5.1. **Proof of the main theorem.**

The proof of free case is the most typical, we list it below:

Free case: In a W^*-probability space $(\mathcal{A},\phi)$ such that ϕ is faithful. Let $\{E(n)\}_{n\in\mathbb{N}}$ be a sequence of orthogonal Hopf algebras such that $A_s(n) \subseteq E(n) \subseteq A_o(n)$ for each n. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of random variables which generate A. Suppose that the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $E(n)$ invariant for all n. By Proposition 3.11, $(x_i)_{i\in\mathbb{N}}$ are $A_s(n)$ invariant for all n. By Köstler and Speicher[9], there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ that $(x_i)_{i\in\mathbb{N}}$ are freely independent and identically distributed with respect to E. It proves the statement 1 for free case. In addition, By Proposition 4.3 in [9] and Definition 3.12, the coaction invariant condition for ϕ can be extended to the conditional expectation E, i.e.

$$E[b_0x_{i_1}b_1\cdots b_{k-1}x_{i_k}b_k]\otimes 1_{E(n)}=\sum_{j_1,...,j_k=1}^{n}E[b_0x_{j_1}b_1\cdots b_{k-1}x_{j_k}b_k]\otimes u_{j_1,i_1}\cdots u_{j_k,i_k}$$

for $i_1,...,i_k\leq n$, where $u_{i,j}$'s are generators of $E(n)$.

2. Suppose that $A_s(n)\subseteq E(n)\subseteq A_b(n)$ for all n and there exists a k such that $E(k)\neq A_s(k)$. Let $\{u_{i,j}\}_{i,j=1,...,k}$'s be generators of $E(k)$. By proposition 3.5, $\exists\, i'$ such that

$$\sum_{l=1}^{k}u_{l,i'}^m\neq 1,$$

for all $m>2$.

Without loss of generality, we assume that $i'=1$. In order to finish the proof, we need to show that $\kappa_l(x_1b_1,....,x_1b_l)=0$ for all $l\geq 3$, where $b_1,....,b_l\in B$. We prove this by induction on l. First, we have that

$$\begin{aligned}
&E[x_1b_1\cdots x_1b_l]\otimes 1_{E(n)}\\
=&\sum_{\mathbf{i}\in[k]^l}E[x_{i_1}b_1\cdots x_{i_l}b_l]\otimes u_{\mathbf{i},1}\\
=&\sum_{\mathbf{i}\in[k]^l}\sum_{\pi\in NC(l)}\kappa_\pi(x_{i_1}b_1,...,x_{i_l}b_l)\otimes u_{\mathbf{i},1}\\
=&\sum_{\pi\in NC_b(l)}\sum_{\mathbf{i}\in[k]^l}\kappa_\pi(x_{i_1}b_1,...,x_{i_l}b_l)\otimes u_{\mathbf{i},1}+\sum_{\pi\in NC(l)\setminus NC_b(l)}\sum_{\mathbf{i}\in[k]^l}\kappa_\pi(x_{i_1}b_1,...,x_{i_l}b_l)\otimes u_{\mathbf{i},1}\\
=&\sum_{\pi\in NC_b(l)}\sum_{\substack{\mathbf{i}\in[k]^l\\ \pi\leq\ker\mathbf{i}}}\kappa_\pi(x_{i_1}b_1,...,x_{i_l}b_l)\otimes u_{\mathbf{i},1}+\sum_{\pi\in NC(l)\setminus NC_b(l)}\sum_{\substack{\mathbf{i}\in[k]^l\\ \pi\leq\ker\mathbf{i}}}\kappa_\pi(x_{i_1}b_1,...,x_{i_l}b_l)\otimes u_{\mathbf{i},1}\\
=&\sum_{\pi\in NC_b(l)}\sum_{\substack{\mathbf{i}\in[k]^l\\ \pi\leq\ker\mathbf{i}}}\kappa_\pi(x_1b_1,...,x_1b_l)\otimes u_{\mathbf{i},1}+\sum_{\pi\in NC(l)\setminus NC_b(l)}\sum_{\substack{\mathbf{i}\in[k]^l\\ \pi\leq\ker\mathbf{i}}}\kappa_\pi(x_1b_1,...,x_1b_l)\otimes u_{\mathbf{i},1}\\
=&\sum_{\pi\in NC_b(l)}\kappa_\pi(x_1b_1,...,x_1b_l)\otimes 1_{E(n)}+\sum_{\pi\in NC(l)\setminus NC_b(l)}\sum_{\substack{\mathbf{i}\in[k]^l\\ \pi\leq\ker\mathbf{i}}}\kappa_\pi(x_1b_1,...,x_1b_l)\otimes u_{\mathbf{i},1}.
\end{aligned}$$

The first term of the last equality follows that $E(n)$ is a quotient algebra of $A_b(n)$. On the other hand

$$E[x_1b_1,...,x_1b_l]\otimes 1_{E(n)}=\sum_{\pi\in NC_b(k)}\kappa_\pi(x_1b_1,...,x_1b_l)\otimes 1_{E(n)}+\sum_{\pi\in NC(l)\setminus NC_b(l)}\kappa_\pi(x_1b_1,...,x_1b_l)\otimes 1_{E(n)}.$$

Therefore,

$$\sum_{\pi\in NC(l)\setminus NC_b(l)}\sum_{\substack{\mathbf{i}\in[k]^l\\ \pi\leq\ker\mathbf{i}}}\kappa_\pi(x_1b_1,...,x_1b_l)\otimes u_{\mathbf{i},1}=\sum_{\pi\in NC(l)\setminus NC_b(l)}\kappa_\pi(x_1b_1,...,x_1b_l)\otimes 1_{E(n)}\tag{1}$$

When $l=3$, we have $NC(3)\setminus NC_b(3)=\{1_3\}$, then

$$\sum_{\substack{\mathbf{i}\in[n]^k\\ \pi\leq\ker 1_3}}\kappa_{1_3}(x_1b_1,...,x_1b_3)\otimes u_{\mathbf{i},1}=\kappa_{1_3}(x_1b_1,...,x_1b_3)\otimes 1_{E(n)},$$

which is

$$\kappa_{1_3}(x_1b_1,...,x_1b_k)\otimes(\sum_{l=1}^{k}u_{l,1}^3-1_{E(n)})=0.$$

Therefore, $\kappa_{1_3}(x_1b_1,...,x_1b_3)=0$. Suppose $\kappa_{1_l}(x_1b_1,...,x_1b_l)=0$ for $3\leq l\leq m$, then for $\pi\in NC(m+1)$, $\kappa_\pi(x_{i_1}b_1,...,x_1b_{m+1})=0$ if π contains a block whose size is between 3 and m.

Each partition $\pi \in NC(m+1) \setminus NC_b(m+1)$ contains at least one block whose size is greater than 2. Therefore, for $\pi \in NC(m+1) \setminus NC_b(m+1)$, $\kappa_\pi(x_1b_1, ..., x_1b_k) = 0$ if $\pi \neq 1_{m+1}$. Hence, equation 1 becomes

$$\kappa_{1_{m+1}}(x_1b_1, ..., x_1b_{m+1}) \otimes (\sum_{l=1}^{k} u_{l,1}^{m+1} - 1_{E(n)}) = 0$$

which implies

$$\kappa_{1_{m+1}}(x_1b_1, ..., x_1b_{m+1}) = 0,$$

for all $b_1, ..., b_{m+1} \in \mathcal{B}$. The proof is complete.

3. Suppose that $A_s(n) \subseteq E(n) \subseteq A_h(n)$ for all n and there exists a k such that $E(k) \neq A_s(k)$. Let $\{u_{i,j}\}_{i,j=1,...,k}$'s be generators of $E(k)$. By proposition 3.5, $\exists\, i'$ such that

$$\sum_{l=1}^{k} u_{l,i'}^{m} \neq 1,$$

for all odd numbers m.

Without loss of generality, we assume that $i' = 1$. We need to show that $\kappa_k(x_1b_1, ..., x_1b_l) = 0$ for all add numbers k where $b_1, ..., b_l \in B$. Agian, we prove this by induction on l.
We have that

$$\begin{aligned} & E[x_1b_1 \cdots x_1b_l] \otimes 1_{E(n)} \\ = & \sum_{\pi \in NC_h(l)} \sum_{\substack{\mathbf{i} \in [k]^l \\ \pi \leq \ker \mathbf{i}}} \kappa_\pi(x_1b_1, ..., x_1b_l) \otimes u_{\mathbf{i},1} + \sum_{\pi \in NC(l) \setminus NC_h(l)} \sum_{\substack{\mathbf{i} \in [k]^l \\ \pi \leq \ker \mathbf{i}}} \kappa_\pi(x_1b_1, ..., x_1b_l) \otimes u_{\mathbf{i},1} \\ = & \sum_{\pi \in NC_h(l)} \kappa_\pi(x_1b_1, ..., x_1b_l) \otimes 1_{E(n)} + \sum_{\pi \in NC(l) \setminus NC_h(l)} \sum_{\substack{\mathbf{i} \in [k]^l \\ \pi \leq \ker \mathbf{i}}} \kappa_\pi(x_1b_1, ..., x_1b_l) \otimes u_{\mathbf{i},1} \end{aligned}$$

The first term of the last equality follows that $E(n)$ is a quotient algebra of $A_h(n)$. On the other hand, we have

$$E[x_1b_1, ..., x_1b_l] \otimes 1_{E(n)} = \sum_{\pi \in NC_b(l)} \kappa_\pi(x_1b_1, ..., x_1b_l) \otimes 1_{E(n)} + \sum_{\pi \in NC(l) \setminus NC_h(l)} \kappa_\pi(x_1b_1, ..., x_1b_l) \otimes 1_{E(n)}.$$

Therefore,

$$\sum_{\pi \in NC(l) \setminus NC_b(l)} \sum_{\substack{\mathbf{i} \in [k]^l \\ \pi \leq \ker \mathbf{i}}} \kappa_\pi(x_1b_1, ..., x_1b_l) \otimes u_{\mathbf{i},1} = \sum_{\pi \in NC(l) \setminus NC_b(l)} \kappa_\pi(x_1b_1, ..., x_1b_l) \otimes 1_{E(n)} \tag{2}$$

When $l = 1$, we have $NC(1) \setminus NC_h(1) = \{1_1\}$, then

$$\kappa^{(1)}(x_1b_1) \otimes (\sum_{l=1}^{k} u_{l,1} - 1_{E(n)}) = 0.$$

Therefore, $\kappa_{1_1}(x_1b_1) = 0$. Suppose $\kappa_{1_l}(x_1b_1, ..., x_1b_l) = 0$ for odd numbers $l \leq 2m$, then for $\pi \in NC(2m+1)$, $\kappa_\pi(x_{i_1}b_1, ..., x_1b_{2m+1}) = 0$ if π contains a block whose size is an odd number less than $2m$. Each partition $\pi \in NC(2m+1) \setminus NC_b(2m+1)$ contains at least one block whose size is odd. Therefore, for $\pi \in NC(2m+1) \setminus NC_b(2m+1)$, $\kappa_\pi(x_1b_1, ..., x_1b_{2m+1}) = 0$ if $\pi \neq 1_{2m+1}$. Hence, equation 2 becomes

$$\kappa_{1_{2m+1}}(x_1b_1, ..., x_1b_{2m+1}) \otimes (\sum_{l=1}^{k} u_{l,1}^{2m+1} - 1_{E(n)}) = 0$$

which implies

$$\kappa_{1_{m+1}}(x_1b_1, ..., x_1b_{m+1}) = 0,$$

for all $b_1, ..., b_{m+1} \in \mathcal{B}$. The proof is complete.

4. If there exist k_1, k_2 such that $E(k_1) \nsubseteq A_h(k_1)$ and $E(k_2) \nsubseteq A_b(k_2)$, by Case 3 and 4, the only non-vanishing cumulants are pair partition cumulants. The proof is done.

Classical Case: The proof is almost the same as free case, we just need to replace noncrossing partitions by all partitions.

boolean Case: The proof is a little different. Some properties of boolean conditional expectation are discussed in [11], [7]. As it is shown in [11], for boolean de Finetti theorem, we need to consider random variables in W^*-probability space with a non-degenerated state $(\mathcal{A}, \phi)$. Assume that $\mathcal{A}$ is generated by a sequence of random variables $(x_i)_{i\in\mathbb{N}}$. Let $\{E(n)\}_{n\in\mathbb{N}}$ be a sequence of orthogonal boolean quantum groups such that $B_s(n) \subseteq E(n) \subseteq B_o(n)$ for each n. If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $E(n)$ invariant, then the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $B_s(n)$ invariant for all n. By the main results in [11], there are a W^*-subalgebra(not necessarily contain the unit of $\mathcal{A}$) $\mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that $(x_i)_{i\in\mathbb{N}}$ are boolean independent and identically distributed with respect to E. In this part of proof, we will assume that $\mathcal{B}$ does not contain $1_\mathcal{A}$. It should be pointed out that the case that $\mathcal{B}$ contains the unit of $\mathcal{A}$ is always a unitalization of the case that $\mathcal{B}$ does not contain $1_\mathcal{A}$. Under our assumption, the tail algebra

$$\mathcal{B} = \bigcap_{n=1}^{\infty} W^*\{x_k | k \geq n\},$$

where $W^*\{x_k|k \geq n\}$ is the WOT closure of the non-unital algebra generated by $\{x_k|k \geq n\}$. We call $\mathcal{B}$ the non-unital tail algebra of $\{x_i\}_{i\in\mathbb{N}}$. Unlike the proof of free and classical case, the coaction invariant condition for ϕ can be extended to the conditional expectation E directly. Actually, we have a stronger statement.

Proposition 5.1. *Let $(\mathcal{A}, \phi)$ be a W^*-probability space and $(x_i)_{i\in\mathbb{N}}$ be an infinite sequence of selfadjoint random variables which generate $\mathcal{A}$ as a von Neumann algebra and the unit of $\mathcal{A}$ is contained in the WOT closure of the non-unital algebra generated by $(x_i)_{i\in\mathbb{N}}$. Let $E(n)$ be a sequence of boolean orthogonal quantum semigroups such that $B_s(n) \subset E(n) \subset B_o(n)$. If $(x_i)_{i\in\mathbb{N}}$ is $E(n)$ invariant for all n, then there exists a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is the non-unital tail algebra of $\{x_i\}_{i\in\mathbb{N}}$, such that $(x_i)_{i\in\mathbb{N}}$ is boolean independent with respect to E. Let $\mathcal{A}_n$ be the non-unital algebra generated by $\{x_i\}_{i\in\mathbb{N}}$. We have that*

$$E[a_1ba_2] = E[a_1]bE[a_2],$$

where $a_1, a_2 \in \mathcal{A}_n$ for some n and $b \in \mathcal{B}$. Let $\{u_{i,j}\}_{i,j=1,...,n}$ be generators of $E(n)$. We will have that

$$E[x_{i_1} \cdots x_{i_k}] \otimes \mathbf{P} = \sum_{j_1,...,j_k=1}^{n} E[x_{j_1} \cdots x_{j_k}] \otimes u_{j_1,i_1} \cdots u_{j_k,i_k}\mathbf{P}$$

for $i_1, ..., i_k \leq n$.

Proof. The existence of E is prove in [11]. We will just need to prove the last two equations. Given $a_1, a_2 \in \mathcal{A}_n$ for some n and $b \in \mathcal{B}$, by assumption, b is contained in W^*-closure of the non-unital algebra generated by $\{x_i|i > n\}$. By Kaplansky theorem, $\exists$ a sequence of bounded elements y_i such that y_i is contained in the non-unital algebra generated by $\{x_i|i > n\}$ such that y_i converges to b in strong operator topology. Therefore, by normality of E, we have

$$E[a_1ba_2] = \lim_{i\to\infty} E[a_1y_ia_2] = \lim_{i\to\infty} E[a_1]E[y_i]E[a_2] = E[a_1]bE[a_2],$$

where the second equality follows the fact that $(x_i)_{i\in\mathbb{N}}$ are boolean independent with respect to E. The second equation can be checked pointwisely. Let $a_1, a_2 \in \mathcal{A}_m$ for some m. In [11], we showed that there exists a normal homomorphism $\alpha : \mathcal{A} \to \mathcal{A}$ such that $\alpha(x_i) = x_{i+1}$ for all $i \in \mathbb{N}$. By the proof of Lemma 6.7 in [11] and the assumption that $\{x_i\}_{i\in\mathbb{N}}$ is $E(n)$-invariant, we have

$$\begin{aligned}
& \phi(a_1 E[x_{i_1}\cdots x_{i_k}]a_2)\otimes \mathbf{P} \\
= & \lim_{l\to\infty, l>m} \phi(a_1\alpha^l(x_{i_1}\cdots x_{i_k})a_2)\otimes \mathbf{P} \\
= & \lim_{l\to\infty, l>m} \phi(\alpha^n(a_1)x_{i_1}\cdots x_{i_k}\alpha^n(a_2))\otimes \mathbf{P} \\
= & \lim_{l\to\infty, l>m} (\phi(\alpha^n(a_1)\sum_{j_1,\dots,j_k=1}^{n} x_{j_1}\cdots x_{j_k}\alpha^n(a_2))\otimes u_{j_1,i_1}\cdots u_{j_k,i_k}\mathbf{P} \\
= & \lim_{l\to\infty, l>m} \phi(a_1\alpha^l(\sum_{j_1,\dots,j_k=1}^{n} x_{j_1}\cdots x_{j_k})a_2)\otimes u_{j_1,i_1}\cdots u_{j_k,i_k}\mathbf{P} \\
= & \sum_{j_1,\dots,j_k=1}^{n} \phi(a_1 E[x_{j_1}\cdots x_{j_k}]a_2)\otimes u_{j_1,i_1}\cdots u_{j_k,i_k}\mathbf{P}
\end{aligned}$$

Since a_1, a_2 are arbitrarily from the sense set $\bigcup_{n\to\infty}\mathcal{A}_n$ of $\mathcal{A}$, the proof is done. □

Now, we turn to finish the proof of our main theorem for boolean case:

1. This is just the boolean de Finetti theorem in [11].
2. As the free case, we need to show that $b_E^{(l)}(x_1b_1, ..., x_1b_l) = 0$ for all $l \geq 3$ where $b_1, ..., b_l \in B \cup \{\mathbb{C}1_{\mathcal{A}}\}$. By proposition 5.1, we have

$$\begin{aligned}
& E[x_{ii_1}b_1x_{\mathbf{i}_2}\cdots b_{n-1}x_{\mathbf{i}_m}] \\
= & E[x_{ii_1}]b_1E[x_{\mathbf{i}_2}]\cdots b_{n-1}E[x_{\mathbf{i}_m}] \\
= & \sum_{\pi_1\in I(k_1)} b_E^{(\pi_1)}(x_{i_1^{(1)}},...x_{i_{k_1}^{(1)}})b_1 \sum_{\pi_2\in I(k_2)} b_E^{(\pi_2)}(x_{i_1^{(2)}},...x_{i_{k_2}^{(2)}})\cdots b_{n-1}\sum_{\pi_m\in I(k_m)} b_E^{(\pi_m)}(x_{i_1^{(m)}},...x_{i_{k_m}^{(m)}}) \\
= & \sum_{\pi\in I(k_1)\times I(k_2)\times\cdots\times I(k_m)} b_E^{(\pi)}(x_{i_1^{(1)}},...x_{i_{k_1}^{(1)}}, b_1x_{i_1^{(2)}},...x_{i_{k_2}^{(2)}},\cdots, b_{n-1}x_{i_1^{(m)}},...x_{i_{k_m}^{(m)}})
\end{aligned}$$

where $\mathbf{i}_l = (i_1^{(l)}, ..., i_{k_l}^{(l)}) \in [n]^{k_l}$ for all $l = 1, ..., m$ for some n and $b_1, ..., b_m \in \mathcal{B}$. Therefore, to finish the prove, we just need to show that $b_E^{(k)}(x_1, ..., x_1) = 0$ for all $l \geq 3$. The rest of the poof is almost the same as the free case:

Let $\{u_{i,j}\}_{i,j=1,...,k}$'s and $\mathbf{P}$ be generators of $E(k)$. First, by Proposition 5.1, we have

$$\begin{aligned}
& E[x_1\cdots x_1]\otimes \mathbf{P} \\
= & \sum_{\mathbf{i}\in[k]^l} E[x_{\mathbf{i}}]\otimes u_{\mathbf{i},1}\mathbf{P} \\
= & \sum_{\mathbf{i}\in[k]^l}\sum_{\pi\in I(l)} b_E^{(\pi)}(x_{\mathbf{i}})\otimes u_{\mathbf{i},1} \\
= & \sum_{\pi\in I_b(l)}\sum_{\mathbf{i}\in[k]^l} b_E^{(\pi)}(x_{i_1},...,x_{i_l})\otimes u_{\mathbf{i},1}\mathbf{P} + \sum_{\pi\in I(l)\setminus I_b(l)}\sum_{\mathbf{i}\in[k]^l} b_E^{(\pi)}(x_{i_1},...,x_{i_l})\otimes u_{\mathbf{i},1}\mathbf{P} \\
= & \sum_{\pi\in I_b(l)}\sum_{\substack{\mathbf{i}\in[k]^l\\ \pi\leq\ker\mathbf{i}}} b_E^{(\pi)}(x_{i_1},...,x_{i_l})\otimes u_{\mathbf{i},1}\mathbf{P} + \sum_{\pi\in I(l)\setminus I_b(l)}\sum_{\substack{\mathbf{i}\in[k]^l\\ \pi\leq\ker\mathbf{i}}} b_E^{(\pi)}(x_{i_1},...,x_{i_l})\otimes u_{\mathbf{i},1}\mathbf{P} \\
= & \sum_{\pi\in I_b(l)}\sum_{\substack{\mathbf{i}\in[k]^l\\ \pi\leq\ker\mathbf{i}}} b_E^{(\pi)}(x_1,...,x_1)\otimes u_{\mathbf{i},1}\mathbf{P} + \sum_{\pi\in I(l)\setminus I_b(l)}\sum_{\substack{\mathbf{i}\in[k]^l\\ \pi\leq\ker\mathbf{i}}} b_E^{(\pi)}(x_1,...,x_1)\otimes u_{\mathbf{i},1}\mathbf{P} \\
= & \sum_{\pi\in I_b(l)} b_E^{(\pi)}(x_1b_1,...,x_1b_l)\otimes \mathbf{P} + \sum_{\pi\in I(l)\setminus I_b(l)}\sum_{\substack{\mathbf{i}\in[k]^l\\ \pi\leq\ker\mathbf{i}}} b_E^{(\pi)}(x_1,...,x_1)\otimes u_{\mathbf{i},1}\mathbf{P}.
\end{aligned}$$

The first term of the last equality follows that $E(n)$ is a quotient algebra of $B_b(n)$. On the other hand

$$E[x_1,...,x_1]\otimes \mathbf{P} = \sum_{\pi\in I_b(k)} b_E^{(\pi)}(x_1,...,x_1)\otimes \mathbf{P} + \sum_{\pi\in I(l)\setminus I_b(l)} b_E^{(\pi)}(x_1,...,x_1)\otimes \mathbf{P}.$$

Therefore,

$$\sum_{\pi\in I(l)\setminus I_b(l)} \sum_{\substack{\mathbf{i}\in[k]^l\\ \pi\leq \ker \mathbf{i}}} b_E^{(\pi)}(x_1,...,x_1)\otimes u_{\mathbf{i},1}\mathbf{P} = \sum_{\pi\in I(l)\setminus I_b(l)} b_E^{(\pi)}(x_1,...,x_1)\otimes \mathbf{P} \tag{3}$$

By assumption, $E(k)$ has a quotient algebra $E'(k)$ that $A_s(k)\subsetneq E'(k)\subseteq A_n(n)$. Let $\{u'_{i,j}\}'s$ be the generators of $E'(k)$. Then, there exists a C^*-homomorphism $\Psi: E(k)\to E'(k)$ such that

$$\Psi(u_{i,j}) = u'_{i,j} \text{ for all } i,j=1,...,k, \quad \text{and } \Psi(\mathbf{P}) = 1_{E'(k)}.$$

Without loss of generality, by proposition 3.5, we can assume that

$$\sum_{l=1}^k u_{l,1}^m \neq 1,$$

for all $m>2$. Let $id\otimes\Psi$ acts on equation 4. Then, we get

$$\sum_{\pi\in I(l)\setminus I_b(l)} \sum_{\substack{\mathbf{i}\in[k]^l\\ \pi\leq \ker \mathbf{i}}} b_E^{(\pi)}(x_1,...,x_1)\otimes u'_{\mathbf{i},1} = \sum_{\pi\in I(l)\setminus I_b(l)} b_E^{(\pi)}(x_1,...,x_1)\otimes 1_{E'(k)}. \tag{4}$$

When $l=3$, we have $I(3)\setminus I_b(3)=\{1_3\}$, then

$$\sum_{\substack{\mathbf{i}\in[n]^k\\ \pi\leq \ker 1_3}} b_E^{(3)}(x_1,...,x_1)\otimes u'_{\mathbf{i},1} = b_E^{(3)}(x_1,...,x_1)\otimes 1_{E'(k)},$$

which is

$$\kappa_{1_3}(x_1,...,x_1)\otimes(\sum_{l=1}^k u'^3_{l,1} - 1_{E'(k)}) = 0.$$

Therefore, $b_E^{(3)}(x_1,...,x_1)=0$.
Suppose $b_E^{(l)}(x_1b_1,...,x_1b_l)=0$ for $3\leq l\leq m$. Then, for $\pi\in I(m+1)$, $b_E^{(\pi)}(x_1,...,x_1)=0$ if π contains a block whose size is between 3 and m. Each partition $\pi\in I(m+1)\setminus I_b(m+1)$ contains at least one block whose size is greater than 2. Therefore, for $\pi\in I(m+1)\setminus I_b(m+1)$, $b_E^{(\pi)}(x_1,...,x_1)=0$ if $\pi\neq 1_{m+1}$. Hence, equation 1 becomes

$$b_E^{(m+1)}(x_1,...,x_1)\otimes(\sum_{l=1}^k u'^{m+1}_{l,1} - 1_{E'(k)}) = 0$$

which implies

$$b_E^{(m+1)}(x_1,...,x_1)=0.$$

The proof is complete.

The same, compare to Case 3 and Case 4 in free case, by applying the method in boolean Case 2, we have Case 3 and Case 4 for boolean independence are also true.

5.2. **Application.** Now, we apply our main theorem to noncommutative distributional symmetries associated with $A_{s'}$, $A_{b'}$, $A_{b\#}$, $C_{s'}$, $C_{b'}$, $B_{s'}$, $B_{b'}$. We have

Corollary 5.2. *Let $(\mathcal{A}, \phi)$ be a W^*-probability space and $(x_i)_{i\in\mathbb{N}}$ be a sequence of random variables which generate $\mathcal{A}$.*

- *Classical case:*
  Suppose that $\mathcal{A}$ is commutative and ϕ is faithful. We have
  1. *If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $C_{s'}(n)$ invariant for all $n \in \mathbb{N}$, then there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ $(x_i)_{i\in\mathbb{N}}$ such that $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and have identically symmetric distribution with respect to E.*
  2. *If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $C_{b'}(n)$ invariant for all $n \in \mathbb{N}$, then there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ $(x_i)_{i\in\mathbb{N}}$ such that $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and have centered Gaussian distribution with respect to E.*
- *Free case:*
  Suppose ϕ is faithful. there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that
  1. *If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $A_{s'}(n)$ invariant for all $n \in \mathbb{N}$, then there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that $(x_i)_{i\in\mathbb{N}}$ are freely independent and have identically symmetric distribution with respect to E.*
  2. *If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $A_{b'}(n)$ invariant for all $n \in \mathbb{N}$, then there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that $(x_i)_{i\in\mathbb{N}}$ are freely independent and have centered semicircular distribution with respect to E.*
  3. *If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $A_{b\#}(n)$ invariant for all $n \in \mathbb{N}$, then there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that $(x_i)_{i\in\mathbb{N}}$ are freely independent and have centered semicircular distribution with respect to E.*
- *boolean case:*
  If ϕ is non-degenerated. Let $\{E(n)\}_{n\in\mathbb{N}}$ be a sequence of orthogonal boolean quantum semigroups such that $B_s(n) \subseteq E(n) \subseteq B_o(n)$ for each n. If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $E(n)$ invariant, then there are a W^*-subalgebra(not necessarily contain the unit of $\mathcal{A}$) $\mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that
  1. *If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $B_{s'}(n)$ invariant for all $n \in \mathbb{N}$, then there are a W^*-subalgebra(not necessarily contain the unit of $\mathcal{A}$) $\mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that $(x_i)_{i\in\mathbb{N}}$ are boolean independent and have identically symmetric distribution with respect to E.*
  2. *If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $B_{b'}(n)$ invariant for all $n \in \mathbb{N}$, then there are a W^*-subalgebra(not necessarily contain the unit of $\mathcal{A}$) $\mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and have centered Bernoulli distribution with respect to E.*

Proof. According the diagrams in Section 3 and 4, we have the following:
1. $C_s(n) \subset C_{s'}(n) \subset C_b(n)$ for all n, and $C_s(n) \neq C_{s'}(n)$ for $n > 3$.
2. $C_{b'}(n) \not\subset C_h(n), C_b(n)$ for $n > 3$.
3. $A_s n \subset A_{s'}(n) \subset A_b(n)$ for all n, and $A_s(n) \neq A_{s'}(n)$ for $n > 3$.
4. $A_{b'}(n), A_{b\#}(n) \not\subset A_h(n), A_b(n)$ for $n > 3$.

5. $B_s(n) \subset B_{s'}(n) \subset B_b(n)$ for all n, and $B_s(n) \neq B_{s'}(n)$ for $n > 3$. Moreover $A_{s'}(n)$ is a quotient algebra of $B_{s'(n)}$
6. $A_{b'}$ is a quotient algebra of $B_{b'}(n)$ and $A_{b'}(n) \not\subset A_h(n), A_b(n)$ for $n > 3$.

By Theorem 1.1, we get our desired results.

□

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