DE FINETTI THEOREMS FOR A BOOLEAN ANALOGUE OF EASY QUANTUM GROUPS

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Abstract. Banica, Curran and Speicher have shown de Finetti theorems for classical and free easy quantum groups. A key of the proof is that each easy quantum group is determined by a category of partitions in the sence of a Tannaka-Krein duality. We define a new kind of category of partitions and associated Boolean analogue of easy quantum groups. Then we prove de Finetti type theorems for them which imply conditional Boolean independence and other distributional restrictions. Our result generalizes Liu's de Finetti theorem.

Introduction

In the study of distributional symmetries in probability theory, the symmetric group S_n and the orthogonal group O_n play a central role. By de Finetti theorem and Hewitt-Savage theorem, if the sequence of real random variables has the joint distribution which is stable under the action of S_n (resp. O_n) , then it is conditionally i.i.d. (resp. conditionally i.i.d. and centered Gaussian). See [Kal05] for details.

In noncommutative probability, there are many analogues of the notion of independence. By [Spe97], there exist only three universal independences; the classical independence, the free independence and the Boolean independence. Free probability theory is one of the most developed noncommutative probability theory [VDN92]. The Boolean independence appeared in [Wal73], [SW97].

An easy quantum group is one of Woronowicz's compact matrix pseudo group (See [Wor87]) which are characterized by a tensor category of partitions in the sense of the Tannaka-Krein duality. The groups S_n, O_n and their free analogue quantum groups $A_s(n), A_o(n)$ are easy quantum groups. The notion of free quantum groups appeared in [Wan95], [Wan98]. In [KS09], Köstler and Speicher have shown a free analogue of de Finetti theorems. The free de Finetti theorem states that the symmetry given by $A_s(n)$ induces the conditional free independence. In [BCS12], Banica, Curran and Speicher have given a unified proof of de Finetti theorems in the classical and free probability by using categories of partitions.

In [Liu14], Liu has defined a quantum semigroup (in the sence of [Sol08]) $\mathcal{B}_s(n)$ and has proved a Boolean de Finetti theorem, which states that the symmetry given by $\mathcal{B}_s(n)$ characterizes the conditional Boolean independence.

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The aim of this paper is to find a new kind of categories of partitions and to construct a Boolean analogue of easy quantum groups which generalizes Liu's de Finetti theorem. The relation between the proof of Liu and ours corresponds to that between [KS09] and [BCS12]. The advantage of using partitions lies in the following facts:

(1) It linearizes the action and gives a nice perspevtive.
(2) We can take other symmetries by visualization.

The main difficulty in carrying out this construction is that Boolean independence is a nonunital phenomenon; if A is a $*$-algebra and ψ is a state on A, then there exists no nontrivial pair of unital subalgebras (A_1, A_2) which is boolean independent with respect to ψ. Nonunital embeddings of von Neumann algebras present a more delicate problem.

This paper consists of 5 sections.

Section 1 is devoted to some preliminaries.

In section 2, we introduce a new category of partitions and corresponding Boolean analogue of easy quantum groups. We further give its classification and the relation between free quantum groups and them. Considering some quotient C*-algebra of them, we get free quantum groups $A_s(n), A_o(n), A_b(n)$ and $A_h(n)$.

Section 3 provides a detailed exposition of a Boolean analogue of easy quantum groups, in particular Haar states and their coactions on polynomial algebras.

Section 4 is devoted to study of operator valued Boolean independence with respect to a nonunital embedding of von Neumann algebras. We extend the notion of a conditional expectation with respect to a nonunital embedding.

In section 5, our main results, the Boolean de Finetti type results are stated and proved in Thm.5.8. Let (M, φ) be a pair of a von Neumann algebra and a non-degenerate normal state. Assume M is generated by self-adjoint elements $(x_j)_{j\in\mathbb{N}}$. Let $C(G_n^I), C(G_n^{I_b}), C(G_n^{I_h})$, and $C(G_n^{I_2})$ be quantum semigroups defined in Def.2.6, which are Boolean analogues of free quantum groups $A_s(n), A_b(n), A_h(n)$ and $A_o(n)$, respectively. Then by Thm.5.8, the following hold.

(1) The joint distribution of $(x_j)_{j\in\mathbb{N}}$ is invariant under the coation of $(C(G_n^I))_{n\in\mathbb{N}}$ if and only if $(x_j)_{j\in\mathbb{N}}$ is conditionally Boolean i.i.d. over M_{tail}.
(2) The joint distribution of $(x_j)_{j\in\mathbb{N}}$ is invariant under the coation of $(C(G_n^{I_b}))_{n\in\mathbb{N}}$ if and only if $(x_j)_{j\in\mathbb{N}}$ is conditionally Boolean i.i.d. over M_{tail} and each x_j has a conditional shifted Bernoulli distribution.
(3) The joint distribution of $(x_j)_{j\in\mathbb{N}}$ is invariant under the coation of $(C(G_n^{I_h}))_{n\in\mathbb{N}}$ if and only if $(x_j)_{j\in\mathbb{N}}$ is conditionally Boolean i.i.d. over M_{tail} and every odd moment of each x_j vanishes.
(4) The joint distribution of $(x_j)_{j\in\mathbb{N}}$ is invariant under the coation of $(C(G_n^{I_2}))_{n\in\mathbb{N}}$ if and only if $(x_j)_{j\in\mathbb{N}}$ is conditionally Boolean i.i.d. over M_{tail} and each x_j has a conditional centered Bernoulli distribution.

1. Preliminaries

Let us define nonunital tail von Neumann algebras.

Notation 1.1.

(1) For $n \in \mathbb{N}$, denote by $\mathscr{P}_n^o$ (resp. $\mathscr{P}_\infty^o$) the *-algebra of all nonunital polynomials in noncommutative n-variables $X_1, \ldots, X_n$ (resp. countably infinite many variables $(X_j)_{j\in\mathbb{N}}$).

(2) Let M be a von Neumann algebra. Let $(x_j)_{j\in\mathbb{N}}$ be a sequence of self-adjoint elements in M. Denote by $\mathrm{ev}_x\colon \mathscr{P}_\infty^o \to M$ the evaluation map $\mathrm{ev}_x(X_j) = x_j$. Let us denote by M_{tail} the nonunital tail von Neumann algebra, that is,

$$M_{\mathrm{tail}} := \bigcap_{n=1}^{\infty} \overline{\mathrm{ev}_x(\mathscr{P}_{\geq n}^o)}^{\sigma w},$$

where $\mathscr{P}_{\geq n}^o := \{f \in \mathscr{P}_\infty^o \mid f \text{ is a polynomial in variables } X_j \ (j \geq n)\}$.

We define the notion of conditional expectations for nonunital embeddings.

Notation 1.2.

(1) In this paper, we do not assume that an embedding of $*$-algebras, C*-algebras or von Neumann algebras is unital.

(2) Let $\eta\colon B \hookrightarrow A$ be an embedding of $*$-algebras. A linear map $E\colon A \to B$ is said to be a conditional expectation with respect to η if it satisfies the following conditions:
   (a) $E(x^*x) \geq 0$ for all $x \in A$,
   (b) $E \circ \eta = \mathrm{id}_B$,
   (c) $E(\eta(b)x) = bE(x), E(x\eta(b)) = E(x)b$ for all $b \in B, x \in A$.

   For an embedding of C*-algebras or von Neumann algebras, we say E is a conditional expectation if it is a conditional expectation between $*$-algebras.

Remark 1.3. Suppose $\eta\colon N \hookrightarrow M$ is an embedding of von Neumann algebras with a conditional expectation $E\colon M \to N$. Then E is a unital completely positive map. Similarly as the case of a unital embedding, we can construct the GNS Hilbert N-N bimodule $L^2(M, E)$ for the pair (η, E). E is said to be nondegenerate if corresponding GNS representation is faithful.

Notation 1.4. Let M, η, and E be same as in Rem.1.3. Let $(x_j)_{j\in J}$ be a family of self-adjoint elements in M. We say $(x_j)_{j\in J}$ is identically distributed with respect to E if $E[x_i^k] = E[x_j^k]$ holds for any $i, j \in J$, and $k \in \mathbb{N}$.

Let us define the notion of conditional Boolean independence.

Definition 1.5. Let $\eta\colon N \hookrightarrow M$ be a normal embedding of von Neumann algebras M, N with a normal conditional expectation $E\colon M \to N$.

(1) Let $(M_j)_{j\in J}$ be a family of σ-weakly closed $*$-subalgebras of M. We do not assume $1_M \in M_j$. Suppose that

$$\eta(N)M_j \subseteq M_j, M_j\eta(N) \subseteq M_j.$$

The family $(M_j)_{j\in J}$ is said to be *Boolean independent with respect to* E or *conditionally Boolean independent over* N if it satisfies

$$E[y_1 \cdots y_k] = E[y_1] \cdots E[y_k],$$

whenever $j_1, \dots, j_k \in J, j_1 \neq j_2 \neq \cdots \neq j_k$, $y_l \in M_{j_l}, l = 1, \dots, k$, and $k \in \mathbb{N}$.

(2) Let $(x_j)_{j\in J}$ be a family of self-adjoint elements of M. We denote by $N\langle x_j\rangle^o$ the $*$-algebra of N-coefficient nonunital noncommutative polynomials of x_j. Let M_j be the σ-weak-closure of $N\langle x_j\rangle^o$ and q_j be the unit of M_j.

The family $(x_j)_{j\in J}$ is said to be Boolean independent with respect to E if $(M_j, \eta_j)_{j\in J}$ is Boolean independent with respect to E.

Let us review some notations related with partitions of a set.

Notation 1.6.

(1) A partition of a set S is a decomposition into disjoint, non-empty subsets. Those subsets are called block of the partition. We denote by $P(S)$ the set of all partitions of S.
(2) For a partition π of a set S and $r, s \in S$, we define $r \underset{\pi}{\sim} s$ if r and s belong to a same block of π.
(3) Let S, J be any sets and $\mathbf{j} \in \mathrm{Map}\,(S, J)$. We will denote by $\ker \mathbf{j}$ the partition of S defined as $r \underset{\ker \mathbf{j}}{\sim} s$ if and only if $j(r) = j(s)$.
(4) For $\pi, \sigma \in P(S)$, we write $\pi \leq \sigma$ if each block of π is a subset of some block of σ. $P(S)$ is a poset under the relation $\leq$.

We define the Möbius function. See [NS06] for more details.

Definition 1.7. Let $(P, \leq)$ be a finite poset. The Möbius function $\mu_P\colon \{(\pi, \sigma) \in P^2 \mid \pi \leq \sigma\} \to \mathbb{C}$ is defined by the following relations: for any $\pi, \sigma \in P$ with $\pi \leq \sigma$,

$$\sum_{\substack{\rho\in P \\ \pi\leq\rho\leq\sigma}} \mu_P(\pi, \rho) = \delta(\pi, \sigma),$$

$$\sum_{\substack{\rho\in P \\ \pi\leq\rho\leq\sigma}} \mu_P(\rho, \sigma) = \delta(\pi, \sigma),$$

where if $\pi = \sigma$ then $\delta(\pi, \sigma) = 1$, otherwise, $\delta(\pi, \sigma) = 0$.

The following remark is one of the most important property of the Möbius function to prove de Finetti theorems.

Remark 1.8. Let Q be a subposet of P which is closed under taking an interval, that is, if $\pi, \sigma \in Q, \rho \in P$ and $\pi \leq \rho \leq \sigma$ then $\rho \in Q$.

Then for any $\pi, \sigma \in Q$ with $\pi \leq \sigma$, we have

$$\mu_Q(\pi, \sigma) = \mu_P(\pi, \sigma).$$

2. Categories of interval partitions and Boolean quantum semigroups

In this section, we define a category of interval partitions and the associated notion of Boolean quantum semigroups, which is a Boolean analogue of easy quantum groups.

Definition 2.1. We denote by $P(k,l)$ the set of all partitions of the disjoint union $[k] \sqcup [l]$, where $[k] = \{1,2,\dots,k\}$ for $k \in \mathbb{N}$. Such a partition will be pictured as

$$p = \begin{Bmatrix} 1\dots k \\ \mathcal{P} \\ 1\dots l \end{Bmatrix}$$

where $\mathcal{P}$ is a diagram joining the elements in the same block of the partition. The *tensor product*, *composition* and *involution* of partitions are obtained by horizontal and vertical concatenation and upside-down turning

$$\begin{aligned} p \otimes q &= \{\mathcal{P}\mathcal{Q}\} \\ pq &= \begin{Bmatrix} \mathcal{Q} \\ \mathcal{P} \end{Bmatrix} - \{\text{closed blocks}\} \\ p^* &= \{\mathcal{P}^\frown\} \end{aligned}$$

where $p = \{\mathcal{P}\}$ and $q = \{\mathcal{Q}\}$ are the pictorial representations of p, q. We denote by $P(k)$ the set $P(0,k)$. If there is no confusion we write $\pi \in P$ if $\pi \in P(k,l)$ for some $k, l \in \mathbb{N} \cup \{0\}$.

The Boolean independence is characterized by Boolean cumulants with combinatorics of interval partitions (See Prop.4.10).

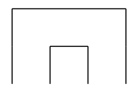
Notation 2.2. $\pi \in P(k)$ is said to be an interval partition of $[k]$ if it contains only consecutive elements. We denote by $I(k)$ the set of all interval partitions of $[k]$. Let $I(k,l) = I(k) \times I(l) \subseteq P(k,l)$.

Remark 2.3. $p \in P(k,l)$ is said to be nondegenerate if its pictorial representation does not have crossing lines. *A category of partitions* (resp. *category of noncrossing partitions*) is a collection of subsets $P_x(k,l) \subseteq P(k,l)$ (resp. $NC_x(k,l) \subseteq NC(k,l)$, $P_x(k,l) \subseteq P(k,l)$), subject to the following conditions from (1) to (6) (resp. from (1) to (5)).

1. It is stable by tensor product.
2. It is stable by composition.
3. It is stable by involution.
4. It contains the unit partition $|$.
5. It contains the duality partition $\sqcap$.
6. It contains the symmetry partition $\slashed{\times}$.

A full category of partitions is a collection of subsets $P_x(k,l) \subseteq P(k,l)$ with the conditions from (1) to (5). The notion of a full category of partition is a generalization of both the category of partition and the noncrossing partitions (See [BC09], Def. 2.2, Def. 3.11, and Def. 6.3 for definitions).

Arbitrary full category of partitions P_x contains the partition



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Therefore, there is no full category of partitions with $P_x \subseteq I$.

Definition 2.4. *A category of interval partitions* is a collection of subsets $I_x(k,l) \subseteq I(k,l)$, subject to the conditions from (1) to (4) in Remark 2.3.

Example 2.5. We denote by $I_h(k), I_2(k), I_b(k) \subseteq I(k)$ the set of all interval partitions with even block size, with block size 2, and with block size ≤ 2 of $[k]$, respectively. Let us denote $I_x(k,l) := I_x(k) \times I_x(l)$ for $k = h, 2$, and b. Then each I_x $(x = h, 2, b)$ is a category of interval partitions.

Let us define a Boolean analogue of easy quantum groups.

Definition 2.6. Denote by $C(G_n^D)$ the universal unital C*-algebra generated by self-adjoint elements $u_{ij}^{(n)} (1 \leq i, j \leq n)$ and an orthogonal projection $p^{(n)}$ with the following relations:

$$\sum_{\substack{\mathbf{i} \in [n]^k \\ \pi \leq \ker \mathbf{i}}} u_{i_1 j_1}^{(n)} \cdots u_{i_k j_k}^{(n)} p^{(n)} = \begin{cases} p^{(n)}, & \pi \leq \ker \mathbf{j}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

$$\sum_{\substack{\mathbf{j} \in [n]^k \\ \pi \leq \ker \mathbf{j}}} u_{i_1 j_1}^{(n)} \cdots u_{i_k j_k}^{(n)} p^{(n)} = \begin{cases} p^{(n)}, & \pi \leq \ker \mathbf{i}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

whenever $\pi \in D(k), k \in \mathbb{N}$. If there is no confusion, we omit the index (n) and just write $u_{i,j}$ and p.

There is a bounded linear map $\Delta: C(G_n^D) \to C(G_n^D) \otimes_{min} C(G_n^D)$ with

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj},$$
$$\Delta(p) = p \otimes p,$$
$$\Delta(1) = 1 \otimes 1.$$

It is easy to check that Δ is a coproduct, that is, the following invariant holds.

$$(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta.$$

Hence $C(G_n^D)$ is a compact quantum semigroup with coproduct Δ (See [Sol08] for the definition of a compact quantum semigroup). We call $C(G_n^D)$ *the Boolean quantum semigroups.*

Remark 2.7. In [Liu14], Liu defined a quantum semigroup $\mathcal{B}_s(n)$ as the universal unital C*-algebra generated by projections $p, u_{i,j} (i,j = 1,\dots,n)$ and relations such that

$$\sum_{j=1}^{n} u_{ij} p = p,$$
$$u_{ij} u_{kj} = 0, \text{ if } k \neq i,$$
$$u_{lj} u_{ij} = 0, \text{ if } l \neq i.$$

There exist a $\star$-homomorphism $\mathcal{B}_s(n)/\langle \sum_{i=1}^n u_{ij} p = p, j = 1,\dots,n\rangle \to C(G_n^I)$ which maps $[u_{ij}]$ to u_{ij} and maps $[p]$ to p.

In [BCS12], Banica, Curran and Speicher have shown de Finetti theorems for easy quantum groups associated with some categories of partitions and of noncrossing partitions. Let us extract their common properties.

Definition 2.8. Let $D = (D(k,l))_{k,l\in\mathbb{N}}$ be a category of partitions (resp. noncrossing partitions, interval partitions).

(D1) D is said to be block-stable if $D(k) = \{\pi \in D(k) \mid \{V\} \in D(|V|) \text{ for any } V \in \pi\}$.

(D2) D is said to be closed under taking an interval if the following hold: if $\rho, \sigma \in D(k)$, $\pi \in P(k)$ (resp. $NC(k)$, $I(k)$), and $\rho \leq \pi \leq \sigma$, then we obtain $\pi \in D(k)$.

We say that D is *blockwise* if it is block-stable and closed under taking an interval. For a blockwise category of partitions D, let us denote

$$L_D := \{k \in \mathbb{N} : \mathbf{1}_l \in D(k)\},$$

where $\mathbf{1}_k \in P(k)$ is the partition which contains only one block $\{1,2,\dots,k\}$. Set $l_D := \sup\{l \in \mathbb{N} \mid 2l \in L_D\}$. If L_D contains some odd number, let us denote $m_D := \sup\{m \in \mathbb{N} \mid 2m-1 \in L_D\}$, and $n_D := \min\{m \in \mathbb{N} \mid 2m-1 \in L_D\}$.

Example 2.9. Let us denote by $\langle \pi_1,\dots,\pi_n\rangle$ the minimal category generated by the unit partition $|$, duality partition $\sqcap$, and the partitions $\pi_1,\dots,\pi_n$ with the categorical operations (i.e. the tensor product, composition and involution).

If D be a block-stable category of (resp. noncrossing) partitions, then L_D is a one of the following 4 sets (see [BC09, Thm.2.6, Thm. 3.14] and [Web13, Prop.2.7] for details):

(1) $L = \{2\}$, producing the category of all pair (resp. noncrossing) partitions $P_2 = \langle \diagup\!\!\!\diagdown \rangle \subseteq P$ (resp. $NC_2 = \langle \varnothing \rangle \subseteq NC$). It correspond to the group O_n (resp. the quantum group $A_o(n)$).

(2) $L = \{1,2,3,\dots\}$, producing the category of all (resp. noncrossing) partitions $P = \langle \uparrow, \sqcap\!\sqcap\!\sqcap, \diagup\!\!\!\diagdown \rangle$ (resp. $NC = \langle \uparrow, \sqcap\!\sqcap\!\sqcap \rangle$). It correspond to the group S_n (resp. the quantum group $A_s(n)$).

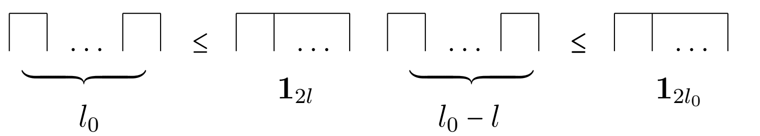
(3) $L = \{2, 4, 6, \ldots\}$, producing the category of all (resp. noncrossing) partitions with blocks of even size $P_h = \langle \sqcap\!\sqcap\!\sqcap, \times \rangle \subseteq P$ (resp. $NC_h = \langle \sqcap\!\sqcap\!\sqcap \rangle \subseteq NC$). It correspond to the group H_n (resp. the quantum group $A_h(n)$).
(4) $L = \{1, 2\}$, producing the category of all (resp. noncrossing) partitions with block size one or two $P_b = \langle \uparrow, \times \rangle \subseteq P$(resp. $NC_b = \langle \uparrow \rangle \subseteq NC$). It correspond to the group B_n (resp. the quantum group $A_b(n)$).

Moreover, these 8 categories are closed under taking interval, thus automatically they are blockwise.

In [BCS12], de Finetti theorems have been proved for these 8 blockwise categories. To find out de Finetti theorems for Boolean independence, let us focus on the blockwise category of interval partitions. Contrary to the classical and the free cases, there exist infinitely many blockwise categories of interval partitions. To see this, we prepare two lemmas.

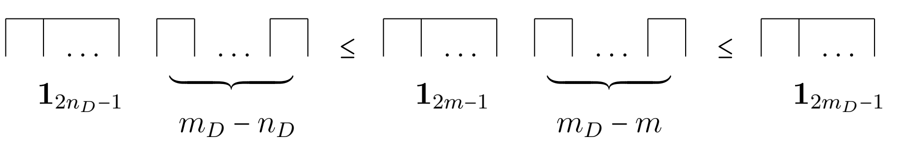
Lemma 2.10. *Let D be a blockwise category of interval partitions. Then the following hold.*
(1) $\{2, 4, \ldots, 2l_D\} \subseteq L_D$.
(2) $\{2n_D - 1, 2n_D + 1, \ldots, 2m_D - 1\} \subseteq L_D$.
(3) $m_D - n_D \leq l_D$.
(4) $l_D < m_D + n_D$ *if* $m_D < +\infty$.

Proof. Pick $l_0 \in \mathbb{N}$ such that $2l_0 \in L_D$. For any $l \in \mathbb{N}, l \leq l_0$,

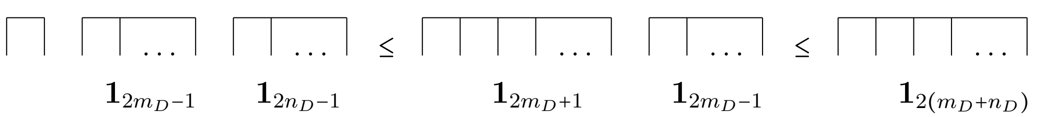


By condition (D2), $\mathbf{1}_{2l} \otimes \sqcap^{\otimes(l_0 - l)} \in D(2l_0)$. By condition (D1), we have $2l \in L_D$, which proves (1).

Assume L_D contains some odd number. We have the following inequalities among partitions.



Hence we obtain $m \in L_D$, which proves (2). To prove (3), let $m \in \mathbb{N}$ with $m \leq m_D$. We can prove that $2(m - n_D) \in L_D$ by the similar discussion as in (1). Hence $m - n_D \leq l_D$. Taking the supremum, we get (3). Assume $m_D < +\infty$. Then



Because $2m_D + 1 \notin L_D$, we obtain $2(m_D + n_D) \notin L_D$, which proves (4).

□

Lemma 2.11. *Let $l_0, m_0, n_0 \in \mathbb{N}$ with $n_0 \leq m_0,\ m_0 - n_0 \leq l_0 < m_0 + n_0$. Define subsets of $\mathbb{N}$ as follows.*

(1) $L(\infty; \infty, n_0) := \{2, 4, 6, \dots\} \cup \{2n_0 - 1, 2n_0 + 1, \dots\}$.
(2) $L(l_0; m_0, n_0) := \{2, 4, 6, \dots, 2l_0\} \cup \{2n_0 - 1, 2n_0 + 1, \dots, 2m_0 - 1\}$.
(3) $L(\infty) := \{2, 4, 6, \dots\}$.
(4) $L(l_0) := \{2, 4, 6, \dots, 2l_0\}$.

Let L be one of 4 sets. For $k \in \mathbb{N}$, assume there are interval partitions ρ, π, π' and σ satisfying following conditions.

(1) $|V|, |V'|, |W|, |X| \in L$ *for any blocks* $V \in \pi, V' \in \pi', W \in \rho$, *and* $X \in \sigma$.
(2) $\rho \leq \pi \otimes \mathbf{1}_k \otimes \pi' \leq \sigma$.

Then we obtain $k \in L$.

Proof. Fix $k \in \mathbb{N}$. In the cases $L = L(\infty), L(l_0)$, it is easy to check $k \in L$ since k must be even.

Assume L contain some odd number. As $\rho \leq \pi \otimes \mathbf{1}_k$, there are blocks $W_1, \dots, W_s \in \rho$ such that $W_1 \cup \dots \cup W_r = \mathbf{1}_k$. If k is odd number, then at least one of $W_1, \dots, W_s$ is an odd block. Hence $k \geq 2n_0 - 1$ if k is an odd number. In particular, we have proved the lemma for $L = L(\infty; \infty, n_0)$.

Hence we only need to consider the case $L = L(l_0; m_0, n_0)$. By $\pi \otimes \mathbf{1}_k \otimes \pi' \leq \sigma$, there are $V_1, \dots V_p, V_1', \dots, V_{p'}' \in \pi$ and $X \in \sigma$ with

$$V_1 \cup \dots \cup V_p \cup \mathbf{1}_k \cup V_1' \cup \dots \cup V_{p'}' = X.$$

Then we have $k < |X| \in L$. Hence if k and $|X|$ have the same parity, $k < 2m_0 - 1$ if k is odd, and $k < 2l_0$ if k is even.

Assume k and $|X|$ have the different parity. Then at least one of $V_1, \dots, V_p, V_1', \dots, V_{p'}'$ is an odd block. Thus we get $k + (2n_0 - 1) \leq 2l_0$ if k is odd and $k < 2m_0 -$ if k is even. By $m_0 - n_0 \leq l_0 < m_0 + n_0$, we have $k \leq 2m_0 - 1$ if k is odd and $k < 2l_0$ if k is even. Hence $k \in L(l_0; m_0, n_0)$. ☐

Example 2.12. We have $L_I = L(\infty; \infty, 1), L_{I_b} = L(1; 1, 1), L_{I_2} = L(1)$, and $L_{I_h} = L(\infty)$.

Now we have the classification of blockwise categories of interval partitions.

Theorem 2.13. *Fix $l_0, m_0, n_0 \in \mathbb{N}$ which satisfy the inequalities in Lemma 2.11. For each $L = L(\infty), L(l_0), L(\infty; \infty, n_0)$ and $L(l_0; m_0, n_0)$, there exists a unique blockwise category D of interval partitions with $L_D = L$. Conversely, for any blockwise category D of an interval partition, we have*

$$L_D = \begin{cases} L(l_D), & \text{if } L_D \text{ contains no odd number.} \\ L(l_D; m_D, n_D), & \text{otherwise.} \end{cases} \tag{2.3}$$

In particular, there exist infinitely many distinct blockwise categories of interval partitions.

Proof. For each $L = L(\infty), L(l_0), L(\infty; \infty, n_0)$ and $L(l_0; m_0, n_0)$, set

$$D(k) := \{\pi \in I(k) \mid |V| \in L \text{ for any } V \in \pi\},\ \ k \in \mathbb{N},$$

respectively. Then D satisfies condition (D1), (D3).

We prove condition (D2). Let $\pi \in I$ and $\rho, \sigma \in D$ such that $\rho \leq \pi \leq \sigma$. Pick any block $\mathbf{1}_k \in \pi$. Then there are $\pi', \pi'' \in I$ with $\pi = \pi' \otimes \mathbf{1}_k \otimes \pi''$.

As $\rho \leq \pi$, there are $\rho_1, \rho_2, \rho_3 \in D$ satisfying

$$\rho = \rho_1 \otimes \rho_2 \otimes \rho_3,$$

$$\rho_1 \leq \pi',\ \ \rho_2 \leq \mathbf{1}_k,\ \ \rho_3 \leq \pi''.$$

We have $\rho \leq \rho_1 \otimes \mathbf{1}_k \otimes \rho_3 \leq \sigma$. By Lemma 2.11 , we get $k \in L$. Since we took an arbitrary block of π, thus $\pi \in D$. Therefore, we have proved (D2).

Conversely, let D be any blockwise category of interval partitions. Then (2.3) holds directly by Lemma 2.10. ☐

By Theorem 2.13, we can shorten the definition of $\{C(G_n^D)\}_{n\in\mathbb{N}}$.

Corollary 2.14. *Let D be a blockwise category of interval partitions and $n \in \mathbb{N}$. Then $C(G_n^D)$ is generated by self-adjoint elements $u_{ij} (1 \leq i,j \leq n)$ and an orthogonal projection p with the following relations:*

for any $k \in L_D$ and $\mathbf{i}, \mathbf{j} \in [n]^k$,

$$\sum_{i=1}^{n} u_{ij_1} \cdots u_{ij_k} p = \begin{cases} p, & j_1 = \cdots = j_k, \\ 0, & \textit{otherwise}, \end{cases}$$

$$\sum_{j=1}^{n} u_{i_1 j} \cdots u_{i_k j} p = \begin{cases} p, & i_1 = \cdots = i_k, \\ 0, & \textit{otherwise}. \end{cases}$$

Corollary 2.15. *Let D be a blockwise category of interval partitions. The quotient C^*-algebra $C(G_n^D)/ < p = 1 >$ is isomorphic to one of the following free quantum groups:*

(1) *$A_o(n)$ if $L_D = L(1)$.*
(2) *$A_s(n)$ if $L_D = L(l_0; m_0, n_0)$ with $(l_0; m_0, n_0) \neq (1; 1, 1)$.*
(3) *$A_h(n)$ if $L_D = L(l_0)$ with $l_0 \geq 2$.*
(4) *$A_b(n)$ if $L_D = L(1; 1, 1)$.*

Proof. For a subset $L \subseteq \mathbb{N}$, denote by J_L the closed ideal of $A_o(n)$ generated by the following relations: for $k \in L$ and $\mathbf{i}, \mathbf{j} \in [n]^k$,

$$\sum_{i=1}^{n} u_{ij_1} \cdots u_{ij_k} = \begin{cases} 1, & j_1 = \cdots = j_k, \\ 0, & \text{otherwise}, \end{cases}$$

$$\sum_{j=1}^{n} u_{i_1 j} \cdots u_{i_k j} = \begin{cases} 1, & i_1 = \cdots = i_k, \\ 0, & \text{otherwise}. \end{cases}$$

Let D be a blockwise category of interval partitions. Then we obtain the isomorphism between the quotient C*-algebras $C(G_n^D)/\langle p = 1 \rangle = A_o(n)/J_{L_D}$. By the same proof of [BC09, Thm.2.6, Thm. 3.14], we have $J_{L_D} = J_L$ where L is one of the 4 sets in Example 2.9, which proves the corollary. □

3. Haar functionals on Boolean quatnum semigroups

The quantum semigroup $C(G_n^D)$ is not a compact quantum group in general. It is known that not every quantum semigroup admits a Haar state. Instead of a Haar state, we construct a linear functional with an invariant property on a subspace of $C(G_n^D)$.

Notation 3.1. We denote by S_n^D the subspace of $C(G_n^D)$ linearly generated by the set

$$\{p\} \cup \{p u_{i_1 j_1} \cdots u_{i_k j_k} p \mid \mathbf{i}, \mathbf{j} \in [n]^k, k \in \mathbb{N}\}.$$

Then S_n^D is a coalgebra with the coproduct Δ.

(1) Fix a complete orthonormal basis $\{e_i\}_{i \in [n]}$ of the n-dimensional Hilbert space l_2^n. We denote by $\Lambda_n^{(k)}$ the linear map $l_2^{n \otimes k} \to l_2^{n \otimes k} \otimes S_n^D$ defined by

$$\Lambda_n^k (e_{j_1} \otimes \cdots \otimes e_{j_k}) := \sum_{\mathbf{i} \in [n]^k} e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes p u_{i_1 j_1} \cdots u_{i_k j_k} p.$$

By direct calculation, Λ_n^k is a linear coaction of S_n^D, that is,

$$(\mathrm{id} \otimes \Delta) \Lambda_n^k = (\Lambda_n^k \otimes \mathrm{id}) \Lambda_n^k.$$

(2) Let $\mathrm{Fix}(\Lambda_n^k)$ denote the invariant subspace of the coaction Λ_n^k, that is,

$$\mathrm{Fix}(\Lambda_n^k) := \{\xi \in l_2^{n \otimes k} \mid \Lambda_n^k(\xi) = \xi \otimes p\}.$$

Denote by $Q^{(k)}$ the orthogonal projection onto $\mathrm{Fix}(\Lambda_n^k)$. For $\mathbf{i}, \mathbf{j} \in [n]^k$, set

$$Q^{(k)}_{\mathbf{i},\mathbf{j}} := \langle e_{i_1} \otimes \cdots \otimes e_{i_k}, Q^{(k)} (e_{j_1} \otimes \cdots \otimes e_{j_k}) \rangle.$$

(3) For $k, n \in \mathbb{N}$ and $\pi \in D(k)$, let

$$T_\pi^{(n)} := \sum_{\substack{\mathbf{j} \in [n]^k \\ \pi \leq \ker \mathbf{j}}} e_{j_1} \otimes \cdots \otimes e_{j_k}.$$

We omit the index (n) if there is no confusion. As D is closed under taking the tensor product, we obtain the equation

$$\mathrm{Fix}(\Lambda_n^k) = \mathrm{Span}\{T_\pi : \pi \in D(k)\}, \tag{3.1}$$

for $k, n \in \mathbb{N}$.

Proposition 3.2 (The Haar Functionals)**.** *For any $n \in \mathbb{N}$, there exists a unique linear functional h on S_n^D satisfying following conditions:*

1. *$h(p) = 1$ and $h(pu_{i_1 j_1} \cdots u_{i_k j_k} p) = Q_{\mathbf{i},\mathbf{j}}^{(k)}$ for $\mathbf{i}, \mathbf{j} \in [n]^k$, and $k \in \mathbb{N}$.*
2. *It has the following invariant property:*

$$(\mathrm{id} \otimes h)\Delta = (h \otimes \mathrm{id})\Delta = h.$$

We call h *the Haar functional* on S_n^D.

Proof. For any $\pi \in D(k)$ and $\mathbf{i} \in [n]^l$,

$$\Lambda_n^{k+l}(T_\pi \otimes e_{i_1} \otimes \cdots \otimes e_{i_l}) = T_\pi \otimes \Lambda_n^{(l)}(e_{i_1} \otimes \cdots \otimes e_{i_l}).$$

Hence

$$Q^{(k+l)}(T_\pi \otimes e_{i_1} \otimes \cdots \otimes e_{i_l}) = T_\pi \otimes Q^{(l)}(e_{i_1} \otimes \cdots \otimes e_{i_l}).$$

Therefore, h is well-defined on S_n^D. The proof of (2) is immediate. □

Remark 3.3 (The Weingarten function)**.** For $\pi, \sigma \in P(k)$, let

$$G_{k,n}(\pi, \sigma) := \langle T_\pi^{(n)}, T_\sigma^{(n)} \rangle.$$

Since the family $(T_\pi^{(n)})_{\pi \in D(k)}$ is linearly independent for large n, $G_{k,n}$ is invertible with respect to the convolution product of $D(k)$ for large n. We define the Weingarten function $W_{k,n}^D$ to be its inverse. Let $Q^{(k)}$ be the orthogonal projection onto $\mathrm{Fix}(\Lambda_n^k)$. Then we have

$$Q_{\mathbf{i},\mathbf{j}}^{(k)} = \sum_{\substack{\pi,\sigma \in D(k) \\ \pi \leq \ker \mathbf{i} \\ \sigma \leq \ker \mathbf{j}}} W_{k,n}^D(\pi, \sigma),$$

for any $\mathbf{i}, \mathbf{j} \in [n]^k$ and sufficiently large n (see [BCS12] for more details).

Since the subposet $D(k) \subseteq I(k)$ is closed under taking an interval (see the condition (D2)), we have $\mu_{I(k)} = \mu_{D(k)}$ by Rem.1.7. By [BCS12, Prop.3.4], we have the following estimate.

Proposition 3.4 (The Weingarten estimate)**.** *For any $\pi, \sigma \in D(k)$,*

$$n^{|\pi|} W_{k,n}^D(\pi, \sigma) = \mu_{I(k)}(\pi, \sigma) + O(\frac{1}{n}) \ (as\ n \to \infty),$$

where we extend the Mobius function by $\mu_{I(k)}(\pi, \sigma) = 0$ when $\pi \nleq \sigma$.

Next we consider the coaction of S_n^D on the $\star$-algebra of nonunital polynomials in noncommutative variables.

Notation 3.5.

(1) For $m, n \in \mathbb{N}$, define a $*$- homomorphism $r_{nm}: C(G_m^D) \to C(G_n^D)$ by

$$r_{nm}(u_{ij}^{(m)}) := \begin{cases} u_{ij}^{(n)}, & i,j \le n, \\ \delta_{ij} 1_{C(G_n^D)}, & \text{otherwise}, \end{cases}$$
$$r_{nm}(p^{(m)}) := p^{(n)}.$$

(2) Define a linear map $\Lambda_n: \mathscr{P}_n^o \to \mathscr{P}_n^o \otimes S_n^D$ by

$$\Lambda_n(X_{j_1} \cdots X_{j_k}) := \sum_{\mathbf{i} \in [n]^k} X_{i_1} \cdots X_{i_k} \otimes p u_{i_1 j_1} \cdots u_{i_k j_k} p.$$

We define a linear map $\Psi_n: \mathscr{P}_\infty^o \to \mathscr{P}_\infty^o \otimes S_n^D$ by

$$\Psi_n(f) := (\mathrm{id} \otimes r_{nm}) \circ \Lambda_m(f),$$

for $f \in \mathscr{P}_m^o \subseteq \mathscr{P}_\infty^o$. Then by direct calculation, each Ψ_n is a linear coaction of S_n^D on $\mathscr{P}_\infty^o$, that is,

$$(\Psi_n \otimes \mathrm{id}) \circ \Psi_n = (\mathrm{id} \otimes \Delta) \circ \Psi_n.$$

See Notation 1.1 for definitions of $\mathscr{P}_n^o$ and $\mathscr{P}_\infty^o$.

(3) Denote by $\mathscr{P}^{\Psi_n}$ the fixed point algebra of the coaction Ψ_n, that is,

$$\mathscr{P}^{\Psi_n} := \{ f \in \mathscr{P}_\infty^o \mid f = f \otimes p \}.$$

(4) Define a linear map $E_n: \mathscr{P}_\infty^o \to \mathscr{P}_\infty^o$ by $E_n := (\mathrm{id} \otimes h) \circ \Psi_n$.

Proposition 3.6. *The following hold:*

(1) Ψ_n *is* $\mathscr{P}^{\Psi_n}$-$\mathscr{P}^{\Psi_n}$ *bilinear map : for each* $f, h \in \mathscr{P}^{\Psi_n}, g \in \mathscr{P}$,

$$\Psi_n(fg) = (f \otimes \mathrm{id}) \Psi_n(g), \ \ \Psi_n(gf) = \Psi_n(g)(f \otimes \mathrm{id}).$$

(2) E_n *is a conditional expectation with respect to the embedding* $\mathscr{P}^{\Psi_n} \hookrightarrow \mathscr{P}_\infty^o$ *(see Definition 1.2).*

Proof. By (3.1), it follows that $\mathscr{P}^{\Psi_n} = \mathrm{Span}\{ f_\pi \in \mathscr{P} \mid \pi \in D(k), k \in \mathbb{N} \}$, where

$$f_\pi := \sum_{\substack{\mathbf{j} \in [n]^k \\ \pi \le \ker \mathbf{j}}} X_{j_1} \cdots X_{j_k}.$$

For any $\mathbf{j} \in [n]^k, \pi \in D(l)$ and $k, l \in \mathbb{N}$,

$$\Psi_n(X_{i_1} \cdots X_{i_k} f_\pi) = \Psi_n(X_{i_1} \cdots X_{i_k})(f_\pi \otimes \mathrm{id}),$$

by relations (2.1). By symmetry, we have proved that Ψ_n is a $\mathscr{P}^{\Psi_n}$-$\mathscr{P}^{\Psi_n}$ bilinear map.

Next, we prove that E_n is a conditional expectation. E_n is also $\mathscr{P}^{\Psi_n}$-$\mathscr{P}^{\Psi_n}$ bilinear map since so is Ψ_n. Clearly we have $E_n[f] = (id \otimes h)(f \otimes p) = f$ for any $f \in \mathscr{P}^{\Psi_n}$.

The proof is completed by showing that $\Psi_n \circ E_n = E_n[\cdot] \otimes p$. Let ν be the natural isomorphism $S_n^D \otimes \mathbb{C} \to S_n^D$. Then

$$\begin{aligned}\Psi_n \circ E_n[f] &= (\mathrm{id} \otimes \nu) \circ (\Psi_n \circ \mathrm{id}) \circ (\mathrm{id} \otimes h) \circ \Psi_n \\ &= (\mathrm{id} \otimes \nu) \circ (\mathrm{id} \otimes \mathrm{id} \otimes h) \circ (\Psi_n \otimes \mathrm{id}) \circ \Psi_n.\end{aligned}$$

As Ψ_n is a linear coaction, the right-hand side is equal to

$$(\mathrm{id} \otimes \nu) \circ (\mathrm{id} \otimes \mathrm{id} \otimes h) \circ (\mathrm{id} \otimes \Delta) \circ \Psi_n.$$

By the invariant property of the Haar functional h, this is equal to to

$$(\mathrm{id} \otimes \nu) \circ \iota \circ (\mathrm{id} \otimes h) \circ \Psi_n,$$

where ι is the embedding $\mathscr{P} \otimes \mathbb{C} \hookrightarrow \mathscr{P} \otimes S_n^D \otimes \mathbb{C}$; $\iota(f \otimes \lambda) = f \otimes p \otimes \lambda$. By the easy computation, this is equal to $E_n[\,\cdot\,] \otimes p$. □

In [BCS12], the conditional expectaion onto the tail algebra is approximated by conditional expectations $(E_n)_{n\in\mathbb{N}}$ onto fixed point von Neumann algebras, to prove free de Finetti theorems. In our setting, conditional expectations $(E_n)_{n\in\mathbb{N}}$ is only defined on the $*$-algebra of polynomals. To prove Boolean de Finetti theorems, we need not consider the normal extension of $(E_n)_{n\in\mathbb{N}}$ (see Prop 5.5 for details).

4. Amalgamated Boolean product

As far as the author knows, a construction of Boolean analogue of amalgamated free product of von Neumann algebras has never been at least explicitly given in literature. Hence in this section we construct a Boolean independent family of von Neumann algebras with respect to a nodegenerate conditional expectation.

Let $(M_j)_{j\in J}$ be a family of von Neumann algebras having normal embeddings $(\eta_j)_{j\in J}$ of common σ-finite von Neumann algebra N. Throughout this section we suppose that each embedding has a normal conditional expectation $E_j \colon M_j \to N$ with faithful GNS representation. We will construct the amalgamated boolean product of the family $(M_j, \eta_j, E_j)_{j\in J}$ over N.

Pick a faithful normal state φ on N. Then each pair $(M_j, \varphi \circ E_j)$ has a faithful GNS representation. Using this GNS representation, we consider M_j acting on $L^2(M_j, \varphi \circ E_j)$. Put $H_j = L^2(M_j, \varphi \circ E_j)$ and $H_j^o = H_j \ominus L^2(N, \varphi)$. Define the amalgamated Boolean product of $(H_j)_{j\in J}$ over N by

$$H = L^2(N, \varphi) \oplus \bigoplus_{j\in J} H_j^o.$$

Let $q_j \in B(H)$ be the orthogonal projection onto the closed subspace $L^2(N, \varphi) \oplus H_j^o \subseteq H$. Set an embedding $\iota_j \colon M_j \to B(H)$ by $\iota_j(x) = x q_j$ for each $j \in J$. We define the amalgamated Boolean product of von Neumann algebras over N by

$$M = \Big(\bigcup_{j\in J} \iota_j(M_j)\Big)''.$$

For $i \neq j$, consider the product $q_i q_j$. Then this element is the orthogonal projection onto the closed subspace $L^2(N, \varphi) \subseteq H$ and it is independent of the choice of i, j.

We denote this by e. We construct the embedding $\eta : N \to B(H)$ as follows. For $b \in N$, we define

$$\eta(b) = \pi_N(b)e + \sum_{j\in J} \iota_j(b)(q_j - e),$$

where $\pi_N : N \to B(L^2(N,\varphi))$ is the GNS representation. Note that the sum converges in the strong operator topology. Then the left and right multiplications of $\eta(N)$ on M_j are those of $\eta_j(N)$;

$$\eta(b)\iota_j(y) = \iota_j(\eta_j(b)y),$$
$$\iota_j(y)\eta(b) = \iota_j(y\eta_j(b)).$$

By definition, $e \in \eta(N)'$. Since φ is faithful, the $*$-homomorphism $N \to \eta(N)e; b \to \eta(b)e$ gives a $*$-isomorphism

$$N \cong \eta(N)e.$$

Proposition 4.1. *We have*

$$\eta(N)e = eMe.$$

For $y \in M$, let $E[y] \in N$ be the element uniquely determined by

$$\eta(E[y])e = eye.$$

Then E is the normal conditional expectation with respect to the embedding η such that the following hold.

1. *$E \circ \iota_j = E_j$.*
2. *The family $(\iota_j(M_j))_{j\in J}$ is Boolean independent with respect to E.*
3. *E has the faithful GNS representation.*

Proof. The inclusion $\eta(N)e \subseteq eMe$ is trivial. At first we have

$$e\iota_j(y)e = \eta(E_j[y])e$$

for each $y \in M_j, j \in J$. If $i \neq j$,

$$\iota_i(x)\iota_j(y) = \iota_i(x)q_iq_j\iota_j(y) = \iota_i(x)e\iota_j(y).$$

Therefore for any indices $j_1, j_2, \ldots, j_k \in J$ with $j_1 \neq j_2 \neq \ldots \neq j_k$,

$$\begin{aligned} e\iota_{j_1}(y_1)\cdots\iota_{j_k}(y_k)e &= (e\iota_{j_1}(y_1)e)(e\iota_{j_2}(y_2)e)\ldots(e\iota_{j_n}(y_n)e) \\ &= \eta_{j_1}(E_{j_1}[y_1])\cdots\eta_{j_k}(E_{j_k}[y_k])e \\ &= \eta(E_{j_1}[y_1]\cdots E_{j_k}[y_k])e. \end{aligned}$$

Hence we have $eMe = \eta(N)e$. Moreover,

$$E[\iota_{j_1}(y_1)\cdots\iota_{j_k}(y_k)] = E_{j_1}[y_1]\cdots E_{j_k}[y_k],$$

whenever $j_1, j_2, \ldots, j_k \in J$, $j_1 \neq j_2 \neq \ldots \neq j_k$. Hence E satisfies the condition (i), (ii). It is immediate from the definition of E that E is a normal conditional expectation with respect to η with the faithful GNS representation. □

Definition 4.2. We write

$$\underset{j\in J}{\diamond_N}(H_j, M_j, \eta_j, E_j) = (\underset{j\in J}{\diamond_N} H_j, \underset{j\in J}{\diamond_N} M_j, \underset{j\in J}{\diamond_N} \eta_j, \underset{j\in J}{\diamond_N} E_j) = (H, M, \eta, E),$$

and call it *the amalgamated Boolean product* of $(H_j, M_j, \eta_j, E_j)_{j\in J}$ over N.

Proposition 4.3. *Let $\eta: N \hookrightarrow M$ be a normal embedding of σ-finite von Neumann algebras with a normal conditional expectation E. $(M_j)_{j\in J}$ be a family of σ weakly-closed subalgebras of M. Suppose the following conditions hold:*

1. *$\eta(N)M_j \subseteq M_j$, $M_j\eta(N) \subseteq M_j$.*
2. *E has the faithful GNS representation.*
3. *$(M_j)_{j\in J}$ generates M.*

If $(M_j)_{j\in J}$ is Boolean independent with respect to E, then M is isomorphic to $(\diamond_N)_{j\in J} M_j$.

Example 4.4.

1. Let $N \subseteq M$ be a unital embedding of σ-finite von Neumann algbras with a faithful normal conditional expectation. Let $M_1 = \langle M, e\rangle$ be its basic construction. Then we have
$$M \diamond_N N = M_1.$$
2. Let $(M_j, \varphi_j)_{j\in J}$ be a family of von Neumann algebras and faithful normal states. Consider the unital embeddings $\mathbb{C} \to M_j$. Then we have
$$\underset{j\in J}{\diamond_{\mathbb{C}}} M_j = B(\underset{j\in J}{\diamond_{\mathbb{C}}} L^2(M_j, \varphi_j)).$$

In free probability (resp. operator valued free probability), cumulants (resp. operator valued cumulants) characterize free independence (resp. free independence with amalgamation) [NS06] [Spe98]. We define the operator valued Boolean cumulants. They combinatorially characterize conditional Boolean independence. Single-variate boolean cumulants are defined in [SW97]. As far as the author knows, multivariate boolean cumulants first appeared in [Leh04].

Throughout the rest of this section, we suppose $N \subseteq M$ is an embedding of von Neumann algebras (not necessarily unital) with a normal conditional expectation E.

See [Leh04, Defn. 1.8] for the definition of the exchangeability system. The notion of an operator valued exchangeability system is obtained by replacing the complex field $\mathbb{C}$ by a subalgebra $N \subseteq M$ and the state by the conditional expectation E onto N.

Notation 4.5. Denote by $(\mathcal{U}, \tilde{E})$ the amalgamated Boolean product $((\diamond_N)_{n=1}^{\infty} M,$ $(\diamond_N)_{n=1}^{\infty} E)$ of countably infinite many copies of (M, E). Let us denote $\iota_n: M \hookrightarrow \mathcal{U}$ the inclusion which maps M to its n-th copy in $\mathcal{U}$. Then $\mathcal{B}_a := (\mathcal{U}, \tilde{E}, (\iota_n)_{n\in\mathbb{N}})$ is an operator valued exchangeability system for (M, E). Let us define partitioned mixed moments by

$$E^{\mathcal{B}_a,\pi}[y_1,\dots,y_k] := \tilde{E}[\iota_{j_1}(y_1)\dots\iota_{j_k}(y_k)],$$

where $\mathbf{j} : [k] \to [n]$ with $\ker \mathbf{j} = \pi$. We omit the index $\mathcal{B}_a$ if there is no confusion, and write E^{π}.

Let $(M_j)_{j\in J}$ be a family of σ-weakly closed subalgebras of M(not necessarily $1_M \in M_j$). Suppose that for any $j \in J$,

$$NM_j \subseteq M_j, M_jN \subseteq M_j. \tag{4.1}$$

$(M_j)_{j\in J}$ is said to be $\mathcal{B}_a$-*independent* if the following condition holds: for any $k \in \mathbb{N}$, $y_l \in M_{j_l}$, $j_l \in J$, $l = 1,\dots,k$ and $\pi \in P(k)$,

$$E^{\pi}[y_1,\dots,y_k] = E^{\pi\wedge\ker\mathbf{j}}[y_1,\dots,y_k].$$

Remark 4.6. By the construction of the exchangeability system, it is easy to see that $(M_j)_{j\in J}$ is $\mathcal{B}_a$-independent if and only if it is Boolean independent with respect to E.

We define operator valued cumulants by replacing the state by the conditional expectation.(See [Leh04, Defn. 2.6] for the definition in the case of $\mathbb{C}$-valued noncommutative probability space).

Definition 4.7. Denote by $(K^{\mathcal{B}_a}_{\pi})_{\pi\in P}$ the cumulants given by the operator valued exchangeability system $\mathcal{B}_a$, that is,

$$K^{\mathcal{B}_a}_{\pi}[y_1,\dots,y_k] := \sum_{\substack{\pi\in P(k)\\ \pi\leq\sigma}} E^{\sigma}[y_1,\dots,y_k]\mu_{P(k)}(\pi,\sigma),$$

and we call them *Boolean cumulants* with respect to E. If there is no confusion, we denote by K^E_{π} the cumulant $K^{\mathcal{B}_a}_{\pi}$. Let us denote by K^E_n the cumulant $K^E_{\mathbf{1}_n}$ for $n \in \mathbb{N}$.

Similar to $\mathbb{C}$-valued exchangeability case, corresponding independence can be characterized by vanishing of mixed cumulants.

Theorem 4.8. *Let $(M_j)_{j\in J}$ be a family of σ-weakly closed $*$-subalgebras of M with (4.2). Then $(M_j)_{j\in J}$ is Boolean independent with respect to E if and only if*

$$K^E_{\pi}[y_{j_1}, y_{j_2},\dots,y_{j_k}] = 0,$$

whenever $\pi \nleq \ker\mathbf{j}$, $\mathbf{j} \in J^k, k \in \mathbb{N}$, $y_j \in M_j$, and $j \in J$.

Proof. By the general result of Lehner [Leh04], we can see that the vanishing of mixed cumulants is equivalent to the $\mathcal{B}_a$-independence. By Rem.4.6, that is also equivalent to Boolean independence with respect to E, which proves the theorem. □

Notation 4.9. Let $(S, \leq)$ be a finite totally ordered set and we write $S = \{s_1 < s_2 < \dots < s_n\}$. For a family $(a_s)_{s\in S}$ of elements in M, we denote by $\prod^{\rightarrow}_{s\in S} a_s$ the ordered product $\prod^{\rightarrow}_{s\in S} a_s = a_{s_1}\cdots a_{s_n}$.

We have the following formula of Boolean cumulants.

Proposition 4.10. *For $\pi \in P(k)$, $y_1, \dots, y_k \in M$ and $k \in \mathbb{N}$,*

$$K^E_\pi[y_1, \dots, y_k] = \begin{cases} \prod^{\rightarrow}_{V \in \pi} K^E_{(V)}[y_1, \dots, y_k], & \text{if } \pi \in I(k), \\ 0, & \text{otherwise}, \end{cases} \tag{4.2}$$

where $K^E_{(V)}[y_1, \dots, y_k] := K^E_m[y_{j_1}, \dots, y_{j_m}]$ for $V = \{j_1 < j_2 < \dots < j_m\}$. For an interval partition π and blocks $V, W \in \pi$, we write $V \leq W$ if $k \leq l$ for any $k \in V$ and $l \in W$. The set π is a totally ordered set under the relation $\leq$.

Moreover, we have

$$E[y_1 \cdots y_k] = \sum_{\pi \in I(k)} K^E_\pi[y_1, \dots, y_k],$$

for $y_1, \dots, y_k \in M$ and $k \in \mathbb{N}$.

Proof. For $\pi \in I(k)$, $y_1, \dots, y_k$ and $k \in \mathbb{N}$, it is easy to see that

$$E^\pi[y_1, \dots, y_k] = \prod^{\rightarrow}_{V \in \pi} E[\prod^{\rightarrow}_{j \in V} y_j].$$

Hence by the same proof of [Leh04, Prop.4.10], we have the factorization rule in (4.2). By the same proof as in [Leh04, Prop.4.11], we obtain the rest of (4.2). We obatain $E[y_1 \cdots y_k] = \sum_{\pi \in P(k)} K^E_\pi[y_1, \dots, y_k]$ by the definition of cumulants Defn.4.7, which completes the proof. □

Corollary 4.11. *Then for $\sigma \in I(k), k \in \mathbb{N}$, we have the moments-cumulants formula*

$$K^E_\sigma[y_1, \dots, y_k] = \sum_{\substack{\pi \in I(k) \\ \pi \leq \sigma}} E^{(\pi)}[y_1, \dots, y_k] \mu_{I(k)}(\pi, \sigma). \tag{4.3}$$

Proof. This is followed by Prop.4.10. □

Example 4.12. A self-adjoint element $x \in M$ is said to be centered Bernoulli, shifted Bernoulli with respect to E if for any $b_1, \dots, b_{k-1} \in N \cup \{1_M\}$ and $k \in \mathbb{N}$, the following hold, respectively.

$$E[xb_1xb_2 \cdots b_{k-1}x] = \sum_{\pi \in I_2(k)} K^E_\pi[xb_1, xb_2, \dots, x],$$
$$E[xb_1xb_2 \cdots b_{k-1}x] = \sum_{\pi \in I_b(k)} K^E_\pi[xb_1, xb_2, \dots, x].$$

Suppose $N = \mathbb{C}1_M$ and E is a normal state on M. Let x be a self-adjoint element in M.

(1) The distribution of x with respect to E is the centered Bernoulli distribution if and only if

$$x \sim \frac{\delta_\sigma + \delta_{-\sigma}}{2}$$

for some $\sigma \geq 0$.

(2) The distribution of x is the shifted Bernoulli distribution if and only if

$$x \sim \frac{\alpha\delta_\alpha + \beta\delta_{-\beta}}{\alpha+\beta}$$

for some $\alpha, \beta > 0$.

Corollary 4.13. *Let $(x_j)_{j\in J}$ be a family of self-adjoint elements in M. Then $(x_j)_{j\in J}$ is Boolean independent identically distributed with respect to E if and only if*

$$E[x_{j_1}b_1x_{j_2}b_2\cdots b_{k-1}x_{j_k}] = \sum_{\substack{\pi\in I(k)\\ \pi\le\ker\mathbf{j}}} K_\pi^E[x_1b_1, x_1b_2, \dots, x_1]$$

for any $b_1,\cdots,b_k \in N\cup\{1_M\}$, $\mathbf{j}\in J^k, k\in\mathbb{N}$, and $a_j \in M_j$, and $j\in J$.

Proof. Thm.4.8 and the vanishing of cumulants for non-interval partitions (see Prop.4.10) proves the corollary. □

5. Boolean De Finetti theorems

Let (M,φ) be a pair of von Neumann algebra and normal state with faithful GNS representation and consider an infinite sequence $(x_j)_{j\in\mathbb{N}}$. We may assume $M\subseteq B(H), \Omega\in H$ is the cyclic vector for M, and φ is implemented by Ω. Throughout this section we suppose $\mathrm{ev}_x(\mathscr{P}_\infty^o)$ is σ-weakly dense in M, where ev_x is the evaluation map(see Notation 1.1. for the definition).

Definition 5.1. We say that the joint distribution of $(x_j)_{j\in\mathbb{N}}$ with respect to φ is G^D-invariant if it is invariant under the coactions of $(C(G_n^D))_{n\in\mathbb{N}}$, that is,

$$(\varphi\circ\mathrm{ev}_x\otimes\mathrm{id})\circ\Psi_n = \varphi\circ\mathrm{ev}_x\otimes p$$

holds for any $n\in\mathbb{N}$.

At first we show the purely combinatorial part of Boolean de Finetti theorems. This proposition is a Boolean analogue of [BCS12, Prop 4.3].

Proposition 5.2. *Suppose $(x_j)_{j\in J}$ is Boolean independent identically distributed with respect to some φ-preserving conditional expectation $E\colon M\to N$, and*

$$K_k^E[x_1, x_1, \dots, x_1] = 0,$$

for all $k\notin L_D$. Then its joint distribution with respect to φ is G^D-invariant.

Proof. By the moment cumulant formula and the relations (2.1), for any $\mathbf{j} \in [n]^k$ and $k \in \mathbb{N}$,

$$
\begin{aligned}
&(\varphi \circ \mathrm{ev}_x \otimes \mathrm{id}) \circ \Psi_n(X_{j_1} \cdots X_{j_k}) \\
&= \sum_{\mathbf{i} \in [n]^k} \varphi(x_{i_1} \cdots x_{i_k}) \otimes p u_{i_1 j_1} \cdots u_{i_k j_k} p \\
&= \sum_{\mathbf{i} \in [n]^k} \sum_{\substack{\pi \in D(k) \\ \pi \leq \ker \mathbf{i}}} K_E^{(\pi)}[x_1, \ldots, x_1] \otimes p u_{i_1 j_1} \cdots u_{i_k j_k} p \\
&= \sum_{\pi \in D(k)} K_E^{(\pi)}[x_1, \ldots, x_1] \otimes \sum_{\substack{\mathbf{i} \in [n]^k \\ \pi \leq \ker \mathbf{i}}} p u_{i_1 j_1} \cdots u_{i_k j_k} p \\
&= \sum_{\substack{\pi \in D(k) \\ \pi \leq \ker \mathbf{j}}} K_E^{(\pi)}[x_1, \ldots, x_1] \otimes p \\
&= \varphi \circ \mathrm{ev}_x(X_{j_1} \cdots X_{j_k}) \otimes p.
\end{aligned}
$$

□

Next, we prove that G^D-invariance implies the existence of the normal conditional expectation onto the tail algebras. This construction was motivated by [Liu14].

Lemma 5.3. *Let $e_{\mathrm{tail}} \in B(H)$ be the orthogonal projection onto the closed subspace $\overline{M_{\mathrm{tail}} \Omega}$ (see Notation 1.1 for the definition of M_{tail}). If the joint distribution of $(x_j)_{j \in \mathbb{N}}$ is G^D-invariant, then*

$$E_{\mathrm{tail}}[y] = e_{\mathrm{tail}}\, y\, e_{\mathrm{tail}} \ \ (y \in M)$$

is a normal conditinal expectation onto M_{tail} with a faithful GNS representation, with respect to the embedding $M_{\mathrm{tail}} \subseteq M$. In particular,

$$M_{\mathrm{tail}} = e_{\mathrm{tail}} M e_{\mathrm{tail}}.$$

Proof. We can construct the following representation of $C(G_n^D)$.

Let $v_1, \ldots, v_{2n}$ be the natural complete orthonormal system of the $2n$-dimensional Hilbert space l_{2n}^2. Set $v_k = v_{k+2n}$ for all $k \in \mathbb{Z}$. Denote $P(v)$ by the orthogonal projection onto $\mathbb{C}v$. Let

$$U_{ij} := P(v_{2(i+j)-3} + v_{2(j-i)+2}), \tag{5.1}$$

$$P := P(v_1 + v_2 + \quad + v_{2n}). \tag{5.2}$$

Then $\pi \colon u_{ij} \mapsto U_{ij}, p \mapsto P$ defines a representation of $C(G_n^D)$.

Hence by [Liu14, Lemma 6.5.], the G^D-invariance implies:

$$\varphi(x_{i_1}^{m_1} \cdots x_{i_k}^{m_k}) = \varphi(x_1^{m_1} \cdots x_k^{m_k}), \tag{5.3}$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_k$ and $m_1, \ldots, m_k \in \mathbb{N}$.

Therefore, the same proof as in [Liu14] remains valid for G^D-invariance: the $*$-endomorphism $sh: M \to M$; $sh(x_j) = x_{j+1}$ is well-defined on M,

$$E_{\text{tail}}[y] = \text{w-}\lim_{n\to\infty} sh^n(y)$$

exists for any $y \in M$ and E is a φ-preserving normal conditional expectation onto M_{tail}.

By [Liu14], the equation 5.3 implies that

$$\varphi(y_1 E_{\text{tail}}[y_2] y_3) = \varphi(E_{\text{tail}}[y_1] y_2 E_{\text{tail}}[y_3]),$$

for any $y_1, y_2, y_3 \in M$. As φ has a faithful GNS representation,

$$E_{\text{tail}}[y] = e_{\text{tail}}\, y\, e_{\text{tail}},$$

for any $y \in M$. Hence $e_{\text{tail}} b e_{\text{tail}} = b$ for any $b \in M_{\text{tail}}$. Let q be the unit of M_{tail}. Then

$$q = E_{\text{tail}}[q] = e_{\text{tail}}\, q\, e_{\text{tail}}.$$

As $e_{\text{tail}} 1_M e_{\text{tail}} = e_{\text{tail}}$, e_{tail} is a projection in M_{tail}. Therefore, $q = e_{\text{tail}}$. Thus $M_{\text{tail}} = E_{\text{tail}}(M) = e_{\text{tail}} M e_{\text{tail}}$.

□

Remark 5.4. Suppose the joint distribution of $(x_j)_{j\in\mathbb{N}}$ is G^D-invariant. Then E_n preserves $\varphi \circ \text{ev}_x$. Hence for any $f \in \mathscr{P}^o_\infty$,

$$e_n \text{ev}_x(f) e_n = \text{ev}_x(E_n(f)) e_n,$$

where e_n is the orthogonal projection onto $\overline{\text{ev}_x(\mathscr{P}^{\Psi_n})\Omega}$.

In [BCS12], a noncommutative martingale convergence theorem of cumulants plays an important role in the proof of de Finetti theorems. Since φ is not faithful, we modify this convergence theorem.

Proposition 5.5. *Let $(\mathscr{B}_n)_{n\in\mathbb{N}}$ be a decreasing sequence of $*$-subalgebras of $\mathscr{P}^o_\infty$, and set*

$$B_\infty := \bigcap_{n\in\mathbb{N}} \text{ev}_x(\mathscr{B}_n).$$

We suppose the following conditions:

(1) *There is a $\varphi \circ \text{ev}_x$ preserving conditional expectation $E_n: \mathscr{P}^o_\infty \to \mathscr{B}_n$ for each $n \in \mathbb{N}$.*
(2) $\overline{B_\infty \Omega} = \overline{M_{\text{tail}} \Omega}$.

Let e_n be the orthogonal projection onto $\overline{\text{ev}_x(\mathscr{B}_n)\Omega}$. Then we have

$$s\text{-}\lim_{n\to\infty} \text{ev}_x(E^\pi_n[f_1, \dots, f_k]) e_n = E^{(\pi)}_{\text{tail}}[f_1(x), \dots, f_k(x)],$$
$$s\text{-}\lim_{n\to\infty} \text{ev}_x(K^{E_n}_\pi[f_1, \dots, f_k]) e_n = K^{E_{\text{tail}}}_\pi[f_1(x), \dots, f_k(x)],$$

whenever $\pi \in I(k), k \in \mathbb{N}$, and $f_1, \dots, f_k \in \mathscr{P}^o_\infty$.

Proof. For any $\pi \in I(k)$,

$$\mathrm{ev}_x(E_n^\pi[f_1,\dots,f_k])e_n = \prod_{V\in\pi}^{\rightarrow} e_n E_n^{(V)}[f_1,\dots,f_k]e_n.$$

By condtition (ii), $\text{s-lim}_{n\to\infty} e_n = e_{\mathrm{tail}}$. Hence we have strong convergence of mixed conditional expectations. The strong convergence of mixed cumulants follows from the fact that mixed cumulants are linear combination of mixed conditional expectations. ☐

By the same proof as in [BCS12, Prop.4.7], we have the following lemma.

Lemma 5.6. *Assume the joint distribution of $(x_j)_{j\in J}$ is G^D-invariant. Then for any $\pi \in D(k)$ and sufficiently large n,*

$$E_n^\pi[X_1,\dots,X_1] = \frac{1}{n^{|\pi|}} \sum_{\substack{\mathbf{i}\in[n]^k \\ \pi\le\ker\mathbf{i}}} X_{i_1}X_{i_2}\cdots X_{i_k}.$$

Lemma 5.7. *Let M be a von Neumann algebra. Fix a nonzero projection $e \in M$. Set a conditional expectation $E\colon M \to N = eMe$ by $E(y) = eye$. Then*

(5.4)

$$K_\pi^E[y_1,\cdots,y_lb,y_{l+1},\cdots,y_k] = \begin{cases} K^E_{\pi|_{[1,l]}}[y_1,\dots,y_l]bK^E_{\pi|_{[l+1,k]}}[y_{l+1},\dots,y_k], & k \not\sim_\pi k+1, \\ 0, & \textit{otherwise}, \end{cases}$$

whenever $b \in N, y_1,\dots,y_k \in M, \pi \in I(k)$ and $l < k$.

Proof.

$$K_2^E[y_1b,y_2] = E[y_1by_2] - E[y_1b]E[y_2] = 0.$$

Let $m \ge 3$. Assume 5.4 holds for any π whose blocks have size less than m. Then

$$\begin{aligned} &K_m^E[y_1\cdots,y_lb,y_{l+1},\cdots,y_m] \\ &= E[y_1\dots y_lby_{l+1}\dots y_m] - \sum_{\pi\in I(m),\pi\neq\mathbf{1_m}} K_\pi^E[y_1,\cdots,y_lb,y_{l+1},\cdots,y_m] \\ &= E[y_1\dots y_l]bE[y_{l+1}\dots y_m] - \sum_{\substack{\sigma\in I(l)\\ \rho\in I(m-l)}} K_\sigma^E[y_1,\cdots,y_l]bK_E^{(\rho)}[y_{l+1},\cdots,y_m] \\ &= 0. \end{aligned}$$

Hence if the size of each block in a interval partition π is less than or equal to m, then 5.4 is satisfied. The induction on m proves the lemma. ☐

Now we are prepared to prove our main theorem, de Fintetti theorems for quantum semigroups $C(G^D)$.

Theorem 5.8. *Let (M,φ) be a pair of von Neumann algebra and normal state with faithful GNS representation. Assume M is generated by self-adjoint elements $(x_j)_{j\in\mathbb{N}}$.*

Let D be a blockwise category of interval partitions. If the joint distribution of $(x_j)_{j\in\mathbb{N}}$ is G^D-invariant if and only if the following hold.

(1) *$(x_j)_{j\in\mathbb{N}}$ is Boolean independent identically distributed with respect to E_{tail}.*
(2) *$K_k^{E_{\mathrm{tail}}}[x_1b_1, x_1b_2, \dots, x_1] = 0$ for all $k \notin L_D, b_1, \cdots, b_k \in M_{\mathrm{tail}} \cup \{1\}$.*

In particular, if the above equaivalent conditions are satisfied for one of $D = I_2, I_b$ and I_h, then the following hold:

(a) *x_1 has a conditional centered Bernoulli distribution if $D = I_2$.*
(b) *x_1 has a conditional shifted Bernoulli distribution if $D = I_b$.*
(c) *Every odd moments of x_1 vanish if $D = I_h$.*

Proof. By Prop. 5.2, condition (2) implies condition (1). Assume (1). At first, we prove

$$E_{\mathrm{tail}}[x_{j_1}\cdots x_{j_k}] = \sum_{\substack{\sigma\in D(k)\\ \sigma\le\ker\mathbf{j}}} K_\sigma^{E_{\mathrm{tail}}}[x_1,\dots,x_1], \tag{5.5}$$

for any $j_1,\dots,j_k \in J, k\in\mathbb{N}$. By Remark 3.3 and Lemma 5.6,

$$\begin{aligned}
E_n[X_{j_1}X_{j_2}\cdots X_{j_k}] &= \sum_{\mathbf{i}\in[n]^k} X_{i_1}X_{i_2}\cdots X_{i_k} Q_{\mathbf{ij}}^{(k)}\\
&= \sum_{\mathbf{i}\in[n]^k} X_{i_1}X_{i_2}\cdots X_{i_k} \sum_{\substack{\pi,\sigma\in D(k)\\ \pi\le\ker\mathbf{i},\sigma\le\ker\mathbf{j}}} W_{k,n}(\pi,\sigma)\\
&= \sum_{\substack{\sigma\in D(k)\\ \sigma\le\ker\mathbf{j}}} \sum_{\pi\in D(k)} \Big(\frac{1}{n^{|\pi|}} \sum_{\substack{\mathbf{i}\in[n]^k\\ \pi\le\ker\mathbf{i}}} X_{i_1}X_{i_2}\cdots X_{i_k}\Big) n^{|\pi|} W_{k,n}(\pi,\sigma)\\
&= \sum_{\substack{\sigma\in D(k)\\ \sigma\le\ker\mathbf{j}}} \sum_{\pi\in D(k)} E_n^\pi[X_1,\dots,X_1] n^{|\pi|} W_{k,n}(\pi,\sigma),
\end{aligned}$$

for sufficiently large n. By the Weingarten estimate (3.4) and the moments-cumulants formula (4.3), we have

$$\begin{aligned}
&\|\mathrm{ev}_x(E_n[X_{j_1}X_{j_2}\cdots X_{j_k}])e_n - \sum_{\substack{\sigma\in D(k)\\ \sigma\le\ker\mathbf{j}}} \mathrm{ev}_x(K_\sigma^{E_n}[X_1,\dots,X_1])e_n\|\\
&= \|\mathrm{ev}_x(E_n[X_{j_1}X_{j_2}\cdots X_{j_k}])e_n - \sum_{\substack{\sigma\in D(k)\\ \sigma\le\ker\mathbf{j}}} \sum_{\pi\in D(k)} \mathrm{ev}_x(E_n^\pi[X_1,\dots,X_1])\mu_{I(k)}(\pi,\sigma)e_n\|\\
&\le \frac{C}{n},
\end{aligned}$$

for some constant $C>0$. For any $n_0\in\mathbb{N}$,

$$\|\frac{1}{n^{|\pi|}} \sum_{\mathbf{i}\in[n]^k\setminus[n_0,n]^k} x_{i_1}x_{i_2}\cdots x_{i_k}\| \to 0 \text{ (as } n\to\infty).$$

Thus we have

$$\lim_{n\to\infty} e_n x_{j_1} x_{j_2} \cdots x_{j_k} \Omega = \lim_{n\to\infty} (\mathrm{ev}_x \circ E_n[X_{j_1} X_{j_2} \cdots X_{j_k}]\Omega) \in \overline{M_{\mathrm{tail}}\Omega}.$$

Hence

$$\overline{B_\infty \Omega} = \overline{M_{\mathrm{tail}}\Omega}.$$

By the modified martingale convergence theorem 5.5, we obtain the desired equation (5.5).

The proof is completed by showing that for any $b_0, \dots, b_k \in M_{\mathrm{tail}} \cup \{1\}$, $j_1, \dots, j_k \in J$, and $k \in \mathbb{N}$,

$$E_{\mathrm{tail}}[x_{j_1} b_1 x_{j_2} b_2 \cdots b_{k-1} x_{j_k}] = \sum_{\substack{\sigma \in D(k) \\ \sigma \le \ker \mathbf{j}}} K_\sigma^{E_{\mathrm{tail}}}[x_1 b_1, x_1 b_2, \dots, x_1]. \tag{5.6}$$

We prove this by the induction on $\#\{l \in [k-1]; b_l \neq 1\}$. Pick any $m \in \mathbb{N} \cup \{0\}, m \le k-1$. Assume that (5.6) is proved in the case that $\#\{l \in [k-1]; b_l \neq 1\} < m$. Consider the case $\#\{l \in [k-1]; b_l \neq 1\} = m$. Let $r = \max\{l \in [k-1]; b_l \neq 1\}$. Then by Lemma 5.7 and the condition (D1),

$$\begin{aligned}
&\sum_{\sigma \in D(k), \sigma \le \ker \mathbf{j}} K_\sigma^{E_{\mathrm{tail}}}[x_1 b_1, \dots, x_1 b_r, \dots, x_1] \\
&= \sum_{\substack{\sigma \in D(k), \sigma \le \ker \mathbf{j} \\ r \not\sim_\sigma r+1}} K_{\sigma|_{[1,r]}}^{E_{\mathrm{tail}}}[x_1 b_1, \dots, x_1] b_r K_{\sigma|_{[r+1,k]}}^{E_{\mathrm{tail}}}[x_1 b_{r+1}, \dots, x_1] \\
&= \sum_{\substack{\pi \in D(r) \\ \pi \le \ker \mathbf{j}|_{[1,r]}}} K_\pi^{E_{\mathrm{tail}}}[x_1 b_1, \dots, x_1] b_r \sum_{\substack{\rho \in D(k-r) \\ \rho \le \ker \mathbf{j}|_{[r+1,k]}}} K_\rho^{E_{\mathrm{tail}}}[x_1 b_{r+1}, \dots, x_1] \\
&= E_{\mathrm{tail}}[x_{j_1} b_1 \cdots x_{j_r}] b_r E_{\mathrm{tail}}[x_{j_{r+1}} b_{r+1} \cdots x_{j_k}] \\
&= E_{\mathrm{tail}}[x_{j_1} b_1 x_{j_2} b_2 \cdots b_{k-1} x_{j_k}].
\end{aligned}$$

By the induction on m, (5.6) holds for any $b_0, \dots, b_k \in M_{\mathrm{tail}} \cup \{1\}$, which proves the theorem. ∎

Remark 5.9. The proof of Thm.5.8 has been devided into two steps because there exist $b_1, \dots, b_{k-1} \in \mathscr{P}^{\Psi_n}$ with

$$E_n[X_{j_1} b_1 X_{j_2} b_2 \cdots b_{k-1} X_{j_k}] \neq \sum_{\mathbf{i} \in [n]^k} X_{i_1} b_1 X_{i_2} b_2 \cdots b_{k-1} X_{i_k} \otimes p u_{i_1 j_1} \dots u_{i_k j_k} p,$$

since Ψ_n is not an endomorphism. To see this, consider the case $n = 3$, $k = 2$, $j_1 = j_2 = 3$ and $b_1 = f_{\mathbf{1}_2} \in \mathscr{P}^{\Psi_3}$. Assume

$$E_3[X_3 f_{\mathbf{1}_2} X_3] = \sum_{\mathbf{i} \in [3]^2} X_{i_1} f_{\mathbf{1}_2} X_{i_2} \otimes p u_{i_1,3} u_{i_2,3} p.$$

As the set $\{X_{j_1}X_{j_2}X_{j_3}X_{j_4} \mid j_k = 1,2,3 \text{ and } k = 1,\dots,4\}$ is linearly independent, it follows that

$$pu_{1,3}(\sum_{j=1,2,3} u_{1,j}u_{1,j})u_{3,3}p = pu_{1,3}u_{3,3}p.$$

Let $(U_{i,j}(i,j = 1,2,3), P)$ be the representation (5.1) of $C(G_3^D)$. Then we have

$$PU_{1,3}(\sum_{j=1,2,3} U_{1,j}U_{1,j})U_{3,3}P = \frac{2}{3}P,$$
$$PU_{1,3}U_{3,3}P = 0.$$

We obtain a contradiction.

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