ON THE WELL-POSEDNESS AND SCATTERING FOR THE GROSS-PITAEVSKII HIERARCHY VIA QUANTUM DE FINETTI

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Abstract. We prove the existence of scattering states for the defocusing cubic Gross-Pitaevskii (GP) hierarchy in $\mathbb{R}^3$. Moreover, we show that an exponential energy growth condition commonly used in the well-posedness theory of the GP hierarchy is, in a specific sense, necessary. In fact, we prove that without the latter, there exist initial data for the focusing cubic GP hierarchy for which instantaneous blowup occurs.

1. Introduction

The cubic Gross-Pitaevskii (GP) hierarchy is a system of infinitely many coupled linear PDE's describing a Bose gas of infinitely many particles, interacting via two-body delta interactions (repulsive in the defocusing case, and attractive in the focusing case). It emerges in the derivation of the nonlinear Schrödinger equation (NLS) from a bosonic N-particle Schrödinger system in the limit as $N \to \infty$, where the pair interaction potentials tend to a delta distribution. In this paper, we prove the existence of scattering states for solutions to the defocusing cubic GP hierarchy in $\mathbb{R}^3$. Moreover, we show that an exponential energy growth condition commonly used in the well-posedness theory of the GP hierarchy is, in a specific sense, necessary.

Our approach uses the *quantum de Finetti theorem* as presented in the work of Lewin, Nam and Rougerie [38] (see Section 1.3). We previously applied it in [8] to give a new, short proof of the unconditional uniqueness of solutions to the cubic GP hierarchy in the energy space. The quantum de Finetti theorem allows us to lift a variety of results that hold for the corresponding NLS to the GP hierarchy. In particular, we use this approach in the work at hand to establish the existence of scattering states for the cubic defocusing GP hierarchy in $\mathbb{R}^3$.

Another main goal of this paper is to illuminate an important exponential energy growth condition that is invoked in all works on the well-posedness of the GP hierarchy equations in the literature. We show that if this condition is removed, the focusing GP hierarchy equations become ill-posed. Again, the de Finetti theorem allows us to access this previously elusive problem by relating it to the blowup in H^1 of solutions to the corresponding focusing cubic NLS.

The first derivation of the nonlinear Hartree (NLH) equation from an interacting Bose gas was given by Hepp in [33], via second quantization and coherent states. Lanford, in his fundamental analysis of the $N \to \infty$ limit of N-particle systems in classical mechanics, made central use of the BBGKY hierarchy [36, 37]. The latter was subsequently employed by Spohn for a different derivation of the NLH, in [47]. Fröhlich, Tsai and Yau revisited this topic more recently in [29]. Subsequently, Erdös, Schlein and Yau gave the derivation of the NLS and NLH for a wide range of situations in their landmark works [21, 22, 23, 24]. In their approach, proving the *uniqueness of solutions* to the GP hierarchy in a space of marginal density matrices $L^\infty_{t\in[0,T)}\mathfrak{H}^1$ (defined in (1.7) below) is a crucial ingredient. Their approach involves sophisticated singular integral estimates organized with Feynman graph expansions, and introduces an important combinatorial method that controls the large number of such graphs.

Subsequently, by combining a reformulation of the combinatorial method of [21, 22, 23, 24] with methods from the theory of dispersive PDE's, Klainerman and Machedon [35] gave a shorter proof of uniqueness of solutions in a different solution space, but under the assumption of an a priori condition on the solutions. Their approach was used by various authors for the derivation of the NLS from interacting Bose gases [8, 11, 14, 15, 34, 50, 53]. The analysis of the Cauchy problem for the GP hierarchy was initiated in [9] and continued e.g. in [18, 50].

In [7], we gave a new proof of unconditional uniqueness for solutions to the cubic GP hierarchy in $\mathbb{R}^3$. Our result is equivalent to the uniqueness result in [21, 22, 23, 24]; the proof combines the Erdös-Schlein-Yau combinatorial method [21, 22, 23, 24] in boardgame formulation [35], with an application of the *quantum de Finetti theorem* [38], see Section 1.3.

There exists a variety of different approaches to the derivation of the NLS and NLH from many-body quantum dynamics, due to the contributions of many authors; we refer to [21, 22, 23, 24, 25, 34, 46] and the references therein, and also [1, 5, 26, 27, 28, 32, 31, 33, 44, 45]. These dispersive nonlinear PDE's give a mean field description of the dynamics of Bose-Einstein condensates, whose formation was first experimentally verified in 1995, [6, 16]. For the mathematical study of Bose-Einstein condensation, we refer to [2, 40, 41, 42, 43] and the references therein.

1.1. **Definition of the GP hierarchy.**

The cubic defocusing GP hierarchy on $\mathbb{R}^3$ for an infinite sequence of bosonic marginal density matrices $\Gamma = (\gamma^{(k)})_{k\in\mathbb{N}}$ is defined as the initial value problem

$$i\partial_t\gamma^{(k)} = \sum_{j=1}^{k}[-\Delta_{x_j},\gamma^{(k)}] \,+\, \lambda B_{k+1}\gamma^{(k+1)}$$
$$\gamma^{(k)}(0) = \gamma_0^{(k)}\,, \quad k\in\mathbb{N}\,, \tag{1.1}$$

where $\lambda\in\{1,-1\}$, and where $\gamma^{(k)}(t;\underline{x}_k;\underline{x}'_k)$ is fully symmetric under permutations separately of the components of $\underline{x}_k := (x_1,\dots,x_k)$, and of the components of $\underline{x}'_k := (x'_1,\dots,x'_k)$. We call (1.1) *defocusing* if $\lambda = 1$, and *focusing* if $\lambda = -1$. The interaction term for the k-particle marginal is defined by

$$B_{k+1}\gamma^{(k+1)} \,=\, B^+_{k+1}\gamma^{(k+1)} - B^-_{k+1}\gamma^{(k+1)}\,, \tag{1.2}$$

where

$$B_{k+1}^{+}\gamma^{(k+1)} = \sum_{j=1}^{k} B_{j;k+1}^{+}\gamma^{(k+1)}, \tag{1.3}$$

and

$$B_{k+1}^{-}\gamma^{(k+1)} = \sum_{j=1}^{k} B_{j;k+1}^{-}\gamma^{(k+1)}, \tag{1.4}$$

with

$$\begin{aligned}\Big(B_{j;k+1}^{+}\gamma^{(k+1)}\Big)&(t,x_1,\dots,x_k;x_1',\dots,x_k')\\ &= \int dx_{k+1}dx_{k+1}'\\ &\qquad \delta(x_j-x_{k+1})\delta(x_j-x_{k+1}')\gamma^{(k+1)}(t,x_1,\dots,x_{k+1};x_1',\dots,x_{k+1}')\\ &= \gamma^{(k+1)}(t,x_1,\dots,x_j,\dots,x_k,x_j;x_1',\dots,x_k',x_j),\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}\Big(B_{j;k+1}^{-}\gamma^{(k+1)}\Big)&(t,x_1,\dots,x_k;x_1',\dots,x_k')\\ &= \int dx_{k+1}dx_{k+1}'\\ &\qquad \delta(x_j'-x_{k+1})\delta(x_j'-x_{k+1}')\gamma^{(k+1)}(t,x_1,\dots,x_{k+1};x_1',\dots,x_{k+1}')\\ &= \gamma^{(k+1)}(t,x_1,\dots,x_k,x_j';x_1',\dots,x_j',\dots,x_k',x_j')\,.\end{aligned} \tag{1.6}$$

We say that $B_{j;k+1}^{+}$ *contracts* the triple of variables x_j, x_{k+1}, x_{k+1}', and that $B_{j;k+1}^{-}$ contracts the triple of variables x_j', x_{k+1}, x_{k+1}'.

In [21, 22, 23, 24] and [7], the well-posedness of (1.1) is studied in the space of solutions

$$\mathfrak{H}^1 := \Big\{ (\gamma^{(k)})_{k\in\mathbb{N}} \,\Big|\, \mathrm{Tr}(|S^{(k,1)}[\gamma^{(k)}]|) < R^{2k} \text{ for some constant } R<\infty \Big\} \tag{1.7}$$

where $S^{(k,\alpha)} := \prod_{j=1}^{k}(1-\Delta_{x_j})^{\alpha/2}(1-\Delta_{x_j'})^{\alpha/2}$ for $\alpha>0$.

We write

$$U^{(k)}(t) := \prod_{\ell=1}^{k} e^{it(\Delta_{x_\ell}-\Delta_{x_\ell'})} \tag{1.8}$$

for the free k-particle propagator. A *mild solution* to (1.1) in the space $L^\infty_{t\in[0,T]}\mathfrak{H}^1$ is a sequence of marginal density matrices $\Gamma = (\gamma^{(k)}(t))_{k\in\mathbb{N}}$ solving the integral equation

$$\gamma^{(k)}(t) = U^{(k)}(t)\gamma^{(k)}(0) + i\int_0^t U^{(k)}(t-s)B_{k+1}\gamma^{(k+1)}(s)ds \quad , \quad k\in\mathbb{N}\,, \tag{1.9}$$

satisfying

$$\sup_{t\in[0,T]} \mathrm{Tr}(|S^{(k,1)}[\gamma^{(k)}(t)]|) < R^{2k} \tag{1.10}$$

for a finite constant R independent of k.

1.2. **The cubic NLS.** In the special case of factorized initial data,

$$\gamma_0^{(k)}(\underline{x}_k;\underline{x}'_k) \;=\; \prod_{j=1}^k \phi_0(x_j)\,\overline{\phi_0(x'_j)}\,, \tag{1.11}$$

the condition that $(\gamma_0^{(k)}) \in \mathfrak{H}^1$ implies

$$\mathrm{Tr}(|S^{(k,1)}[\gamma_0^{(k)}]|) = \|\phi_0\|_{H^1}^{2k} < R^{2k} \quad , \quad k \in \mathbb{N}\,, \tag{1.12}$$

and is equivalent to the condition $\|\phi_0\|_{H^1} < R$. A particular solution to (1.1) with initial data (1.11) is given by $\Gamma = (\gamma^{(k)}(t))_{k\in\mathbb{N}}$ where for all $k \in \mathbb{N}$,

$$\gamma^{(k)}(t;\underline{x}_k;\underline{x}'_k) \;=\; \prod_{j=1}^k \phi(t,x_j)\,\overline{\phi(t,x'_j)} \tag{1.13}$$

is factorized. In particular, the 1-particle wave function ϕ satisfies the cubic NLS

$$i\partial_t\phi(t) = -\Delta\phi(t) + \lambda|\phi(t)|^2\phi(t) \quad , \quad \phi(0) = \phi_0 \in H^1\,, \tag{1.14}$$

which is defocusing if $\lambda = 1$ and focusing if $\lambda = -1$.

Solutions to (1.14) conserve the L^2-mass

$$M[\phi(t)] = \|\phi(t)\|_{L_x^2}^2 = M[\phi_0]\,, \tag{1.15}$$

the momentum

$$P[\phi(t)] = i\int \overline{\phi(t,x)}\nabla\phi(t,x)dx\,, \tag{1.16}$$

angular momentum

$$L[\phi(t)] = i\int \overline{\phi(t,x)}\,x\wedge\nabla\phi(t,x)dx\,, \tag{1.17}$$

and the energy

$$E[\phi(t)] = \frac{1}{2}\|\nabla_x\phi(t)\|_{L_x^2}^2 + \frac{\lambda}{4}\|\phi(t)\|_{L_x^4}^4 = E[\phi_0]\,. \tag{1.18}$$

The cubic NLS in $\mathbb{R}^3$ (1.14) is L^2-supercritical and H^1-subcritical, and is globally well-posed in H^1 if $\lambda = 1$, and locally well-posed if $\lambda = -1$, [51].

1.2.1. *The defocusing NLS.* In the defocusing case $\lambda = 1$, (1.14) is globally well-posed and displays the existence of scattering states and asymptotic completeness:

Theorem 1.1. *Let $S_t : \phi_0 \mapsto \phi(t)$ denote the flow map associated to (1.14), for $t \in \mathbb{R}$ and $\lambda = 1$. Then, there exist continuous bijections (wave operators) $W_+, W_- : H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$, such that the strong limit*

$$\lim_{t\to\pm\infty} e^{-it\Delta}S_t(\phi_0) = \phi_\pm \quad , \quad \phi_0 = W_\pm(\phi_\pm) \tag{1.19}$$

holds for all $\phi_0 \in H^1(\mathbb{R}^3)$.

We refer to Section 3.6 in [51] for a detailed discussion and a proof.

1.3. **The quantum de Finetti theorem.** As shown in our recent work [7], solutions to the GP hierarchy and solutions to the NLS are closely interconnected via the *quantum de Finetti theorem*, which is a quantum analogue of the Hewitt-Savage theorem in probability theory, [19]. We quote it in the formulation presented by Lewin, Nam and Rougerie in [38] who coined the notions of the strong and weak quantum de Finetti theorems (here collected into a single theorem).

Theorem 1.2. *Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{H}^k = \bigotimes_{sym}^k \mathcal{H}$ denote the corresponding bosonic k-particle space. Let Γ denote a collection of bosonic density matrices on $\mathcal{H}$, i.e.,*

$$\Gamma = (\gamma^{(1)}, \gamma^{(2)}, \dots)$$

with $\gamma^{(k)}$ a non-negative trace class operator on $\mathcal{H}^k$. Then, the following hold:

- *(Strong Quantum de Finetti theorem, [20, 49, 38]) Assume that Γ is admissible, i.e., $\gamma^{(k)} = \mathrm{Tr}_{k+1}\gamma^{(k+1)}$, where Tr_{k+1} denotes the partial trace over the $(k+1)$-th factor, $\forall k \in \mathbb{N}$. Then, there exists a unique Borel probability measure μ, supported on the unit sphere in $\mathcal{H}$, and invariant under multiplication of $\phi \in \mathcal{H}$ by complex numbers of modulus one, such that*

$$\gamma^{(k)} = \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \quad , \quad \forall k \in \mathbb{N}\,. \tag{1.20}$$

- *(Weak Quantum de Finetti theorem, [38, 3, 4]) Assume that $\gamma_N^{(N)}$ is an arbitrary sequence of mixed states on $\mathcal{H}^N$, $N \in \mathbb{N}$, satisfying $\gamma_N^{(N)} \geq 0$ and $\mathrm{Tr}_{\mathcal{H}^N}(\gamma_N^{(N)}) = 1$, and assume that its k-particle marginals have weak-$*$ limits*

$$\gamma_N^{(k)} := \mathrm{Tr}_{k+1,\cdots,N}(\gamma_N^{(N)}) \;\rightharpoonup^*\; \gamma^{(k)} \quad (N \to \infty)\,, \tag{1.21}$$

in the trace class on $\mathcal{H}^k$ for all $k \geq 1$ (here, $\mathrm{Tr}_{k+1,\cdots,N}(\gamma_N^{(N)})$ denotes the partial trace in the $(k+1)$-st up to N-th component). Then, there exists a unique Borel probability measure μ on the unit ball in $\mathcal{H}$, and invariant under multiplication of $\phi \in \mathcal{H}$ by complex numbers of modulus one, such that (1.20) holds for all $k \geq 0$.

We note that the limiting hierarchies of marginal density matrices obtained via weak-$*$ limits from the BBGKY hierarchy of bosonic N-body Schrödinger systems as in [21, 22, 23, 24] do not necessarily satisfy admissibility.

For the problems considered in this paper, the Hilbert space is given by $\mathcal{H} = L^2(\mathbb{R}^3)$. In [7], we have used Theorem 1.2 to present a new, shorter proof of the *unconditional* uniqueness of solutions to the GP hierarchy in $L^\infty_{t\in[0,T)}\mathfrak{H}^1$; we thereby also obtain a direct correspondence between solutions to the GP hierarchy and solutions to the NLS which will be crucial for our proof of the main results in this paper. The unconditional uniqueness part itself is equivalent to the uniqueness result proven in [21, 22, 23, 24]. Our main result in [7] states the following.

Theorem 1.3. *(Chen-Hainzl-Pavlović-Seiringer, [7]) Let $(\gamma^{(k)}(t))_{k\in\mathbb{N}}$ be a mild solution in $L^\infty_{t\in[0,T)}\mathfrak{H}^1$ to the (de)focusing cubic GP hierarchy in $\mathbb{R}^3$ with initial data $(\gamma^{(k)}(0))_{k\in\mathbb{N}} \in \mathfrak{H}^1$, which is either admissible, or obtained at each t from a weak-$*$ limit as described in Theorem 1.2.*

Then, $(\gamma^{(k)})_{k\in\mathbb{N}}$ *is the unique solution for the given initial data.*

Moreover, assume that the initial data $(\gamma^{(k)}(0))_{k\in\mathbb{N}} \in \mathfrak{H}^1$ *satisfy*

$$\gamma^{(k)}(0) = \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \quad , \quad \forall k \in \mathbb{N}\,, \tag{1.22}$$

where μ *is a Borel probability measure supported either on the unit sphere or on the unit ball in* $L^2(\mathbb{R}^3)$*, and invariant under multiplication of* $\phi \in \mathcal{H}$ *by complex numbers of modulus one. Then,*

$$\gamma^{(k)}(t) = \int d\mu(\phi)(|S_t(\phi)\rangle\langle S_t(\phi)|)^{\otimes k} \quad , \quad \forall k \in \mathbb{N}\,, \tag{1.23}$$

where $S_t : \phi \mapsto \phi(t)$ *is the flow map of the cubic (de)focusing NLS, for* $t \in [0,T)$*. That is,* $\phi(t)$ *satisfies* (1.14) *with initial data* ϕ*. Accordingly,*

$$\gamma^{(k)}(t) = \int d\mu_t(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \quad , \quad \forall k \in \mathbb{N}\,, \tag{1.24}$$

where $d\mu_t(\phi) := d\mu(S_{-t}(\phi))$ *is the push-forward measure under the NLS flow.*

2. Statement of main results

In this paper, we prove the existence of scattering states for the defocusing cubic GP hierarchy in $\mathbb{R}^3$. Moreover, we investigate the necessity of the energy growth condition in the definition of the solution spaces $\mathfrak{H}^1$, see (1.7).

2.1. Scattering for the cubic GP hierarchy in $\mathbb{R}^3$.

We prove the existence of scattering states using the quantum de Finetti theorems, Theorem 1.1, and Theorem 1.3, which was proved in our earlier paper [7]. The initial data for the GP hierarchy $\Gamma_0 = (\gamma_0^{(k)})_{k\in\mathbb{N}}$ have the form

$$\gamma_0^{(k)} = \int d\mu(\phi)\big(|\phi\rangle\langle\phi|\big)^{\otimes k}\,. \tag{2.1}$$

We consider the defocusing cubic NLS with $\lambda = 1$, and assume that

$$\int d\mu(\phi)(E[\phi])^{2k} \leq R^k \tag{2.2}$$

holds for some finite constant $R > 0$, and all $k \in \mathbb{N}$, where

$$E[\phi] = \frac{1}{2}\int |\nabla\phi|^2 dx + \frac{1}{4}\int |\phi|^4 dx\,, \tag{2.3}$$

is the energy functional for the cubic defocusing NLS in $\mathbb{R}^3$. The condition (2.2) is equivalent to μ having support in a ball in H^1; see Lemma 3.1, below.

We note that while the de Finetti theorems provide the existence and uniqueness of a measure μ, μ is in general not explicitly known. Therefore, it is important to express the condition (2.2), directly at the level of density matrices. This is addressed in Section 2.1.1 below, where we review *higher order energy functionals* for GP hierarchies that were first introduced in [10].

The first main result of this paper establishes the existence of scattering states for the cubic defocusing GP hierarchy on $\mathbb{R}^3$, and provides the construction of the corresponding asymptotic measures for the de Finetti representation (1.24). This has been a longstanding open problem despite much activity in the field. With our approach via the de Finetti theorem, it follows from the scattering theory for the NLS.

Theorem 2.1. *Let* $\Gamma_0 = (\gamma_0^{(k)})_{k\in\mathbb{N}}$ *be as in* (2.1)*, and* $\lambda = 1$ *(the defocusing case). We assume that* μ *satisfies* (2.2)*.*

Let $\gamma^{(k)}(t) = \int d\mu(\phi)(|S_t\phi\rangle\langle S_t\phi|)^{\otimes k}$*, for* $k \in \mathbb{N}$*, denote the unique solution to* (1.1) *satisfying* $\gamma^{(k)}(0) = \gamma_0^{(k)}$*, for* $k \in \mathbb{N}$*.*

Then, there exist unique asymptotic measures μ_+*,* μ_- *such that*

$$\gamma_\pm^{(k)} := \int d\mu_\pm(\phi)(|\phi\rangle\langle\phi|)^k \tag{2.4}$$

are scattering states $\gamma_+^{(k)}$*,* $\gamma_-^{(k)}$ *on* $L^2(\mathbb{R}^{3k})$ *satisfying*

$$\lim_{t\to\pm\infty} \mathrm{Tr}\Big(\Big| S^{(k,1)} \Big[U^{(k)}(-t)\gamma^{(k)}(t) - \gamma_\pm^{(k)} \Big] \Big| \Big) = 0 \tag{2.5}$$

for all $k \in \mathbb{N}$*. In particular,*

$$d\mu_\pm(\phi) = d\mu(W_\pm(\phi)) \tag{2.6}$$

where the continuous bijections W_+*,* W_- $: H^1 \to H^1$ *are the wave operators from Theorem 1.1.*

More generally, our method allows to transfer knowledge about the non-linear Schrödinger equation (as given in Theorem 1.1) to results about the GP hierarchy. For instance, if the existence of scattering states for the focusing NLS can be shown for a suitable set of initial data (see for instance [17]), one can also infer a corresponding result for the GP hierarchy for initial states with de Finetti measure μ supported on that set.

2.1.1. *Higher order energy functionals.* The condition on μ given in (2.2) can be formulated directly at the level of marginal density matrices. This is of importance because the initial data for the GP hierarchy is usually provided at the level of density matrices $\gamma_0^{(k)}$, without explicit determination of the measure μ. To this end, we recall the higher order energy functionals that were introduced in [10]. In the case of the cubic GP hierarchy, they are defined by

$$\langle K^{(m)}\rangle_{\Gamma(t)} := \mathrm{Tr}(K^{(m)}\gamma^{(2m)}(t)) \tag{2.7}$$

for $m \in \mathbb{N}$, where

$$\begin{aligned} K_\ell &:= \frac{1}{2}(1-\Delta_{x_\ell})\mathrm{Tr}_{\ell+1} + \frac{1}{4}B^+_{\ell;\ell+1} \quad , \quad \ell \in \mathbb{N}\,, \\ K^{(m)} &:= K_1 K_3 \cdots K_{2m-1}. \end{aligned} \tag{2.8}$$

In [10], it is shown that these higher order energy functionals are conserved.

We note that

$$\mathrm{Tr}(K_1K_3\cdots K_{2k-1}\gamma_0^{(2k)}) = \int d\mu(\phi)(E[\phi])^k \tag{2.9}$$

corresponding to (2.2); see Section 4 of [10].

2.2. **Energy growth condition.** Results on the well-posedness of the Cauchy problem for the GP hierarchy are usually obtained in solution spaces of marginal density matrices where an exponential growth condition either of the form

$$\mathrm{Tr}|S^{(k,1)}[\gamma^{(k)}]| < R^{2k} \quad \forall k \in \mathbb{N} \tag{2.10}$$

holds in the trace norm, or of the form

$$\|S^{(k,1)}[\gamma^{(k)}]\|_{\mathrm{HS}} < R^{2k} \quad \forall k \in \mathbb{N} \tag{2.11}$$

in the Hilbert-Schmidt norm. In the works [8, 9, 10, 11, 12, 13] and [14, 15, 34, 50, 53], well-posedness is studied in solution spaces incorporating the condition (2.11). In [21, 22] and the paper at hand, only the case (2.10) is considered; a condition of this form is an important technical ingredient for these uniqueness proofs. We would like to address the crucial question whether the energy growth condition (2.10) in the definition of the space $\mathfrak{H}^1$ is necessary for a well-posedness theory.

We introduce the quantity

$$\mathcal{R}_{H^1}(\mu) := \exp\Big[\limsup_{k\to\infty} \frac{1}{2k} \log\Big(\int d\mu(\phi)\|\phi\|_{H^1}^{2k} \Big) \Big] , \tag{2.12}$$

which corresponds to the radius of the smallest ball in H^1 that contains the support of μ. We observe that (2.10), expressed via the de Finetti theorem as

$$\int d\mu(\phi)\|\phi\|_{H^1}^{2k} < R^{2k} \quad , \quad \forall k \in \mathbb{N} , \tag{2.13}$$

is equivalent to the condition that μ satisfies

$$\mathcal{R}_{H^1}(\mu) \, < \, R , \tag{2.14}$$

for $R < \infty$. Hence, (2.10) simply means that μ has bounded support in H^1.

Here, we prove that if a faster than exponential growth rate is admitted, so that $\mathcal{R}_{H^1}(\mu) = \infty$, the *focusing* cubic GP hierarchy is *ill-posed*, in the sense that there exist initial data at $t = 0$ for which the solution blows up instantaneously; that is, the norm $\mathrm{Tr}(|S^{(k,1)}\gamma^{(k)}(t)|)$ diverges for any positive $t > 0$.

This result is a consequence of the following well-known result about the blowup in H^1 of solutions of the cubic NLS in the focusing case $\lambda = -1$. Eq. (1.14) is locally well-posed; given any initial data $\phi_0 \in H^1$, there exists $\tau = \tau(\phi_0) > 0$ and a unique solution $\phi(t) \in H^1$ for $t \in [0, \tau)$. However, the solution might only exist for a finite time. Let

$$V[\phi](t) := \|x\phi(t)\|_{L^2}^2 \tag{2.15}$$

denote the quadratic moment in x with respect to $\phi(t)$. Then, blowup in finite time occurs whenever $E[\phi_0] < 0$ and $V[\phi_0] < \infty$. This is proven by use of the *virial identities* (Vlasov-Petrishchev-Talanov [52], Zakharov [54], Glassey [30])

$$\partial_t V[\phi](t) = 2\Im \int x \cdot \overline{\phi(x)} \nabla\phi(x) dx \tag{2.16}$$

and

$$\partial_t^2 V[\phi](t) = 16E[\phi_0] - 2\|\phi(t)\|_{L^4}^4 . \tag{2.17}$$

In fact, if $E[\phi_0] < 0$, the r.h.s. of (2.17) is strictly negative, and therefore, $\|x\phi(t)\|_{L^2}$ tends to zero in finite time. However, by the Heisenberg uncertainty principle,

$$\|\phi_0\|_{L^2}^2 \leq C\|x\phi(t)\|_{L^2}\|\phi(t)\|_{H^1} \,, \tag{2.18}$$

see for instance [51]. Hence, a bound of the form $\|x\phi(t)\|_{L^2} < b(t)$ with $b(t) \searrow 0$ as $t \nearrow T = T(\phi_0)$ implies that the solution blows up in H^1, that is, $\|\phi(t)\|_{H^1} \nearrow \infty$ as $t \nearrow \tau = \tau(\phi_0)$ for some $\tau \leq T$. We refer to $\tau(\phi_0)$ as the blowup time corresponding to the initial data $\phi_0 \in H^1$.

One can easily derive an upper bound on the blowup time as follows. From the virial identity (2.16), it follows that

$$|\partial_t \|x\phi(t)\|_{L^2}^2| \leq 2\|x\phi(t)\|_{L^2}\|\phi(t)\|_{\dot H^1} \,, \tag{2.19}$$

and from (2.17) that

$$\partial_t^2 \|x\phi(t)\|_{L^2}^2 < 16E[\phi(t)] = 16E[\phi] \,, \tag{2.20}$$

where $\phi(t)$ solves the focusing cubic NLS with initial data ϕ. From second order Taylor expansion in t, we thus find that

$$\|x\phi(t)\|_{L^2}^2 \quad \leq \quad \|x\phi\|_{L^2}^2 + 2t\|x\phi\|_{L^2}\|\phi\|_{\dot H^1} + 8t^2 E[\phi] \,. \tag{2.21}$$

While the left hand side is non-negative, the right hand side becomes negative in finite time if $E[\phi] < 0$, which implies that the solution blows up in H^1. If $E[\phi] < 0$, it follows that the quadratic equation on the right hand side has precisely one positive and one negative root. The positive root $T(\phi) > 0$ is an upper bound on the blowup time $\tau(\phi)$.

Combining this with the de Finetti representation (1.23) for solutions to the GP hierarchy, we obtain the following main result.

Theorem 2.2. *Consider the set of probability measures μ on the unit ball in $L^2(\mathbb{R}^3)$. Then, the following dichotomy holds for the focusing cubic GP hierarchy* (1.1) *(where we have $\lambda = -1$):*

- *For the subset of probability measures satisfying*

$$\mathcal{R}_{H^1}(\mu) < \infty \,, \tag{2.22}$$

the following holds. Given $\mu_0 \in \{\mu \,|\, \mathcal{R}_{H^1}(\mu) < \infty\}$, there exists a unique solution to the focusing cubic GP hierarchy in $L^\infty_{[0,T)}\mathfrak{H}^1$, for some $T = T(\mu_0) > 0$, with the initial data

$$\Big(\gamma_0^{(k)} = \int d\mu_0(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \Big)_{k\in\mathbb{N}} \tag{2.23}$$

in $\mathfrak{H}^1$.

- *For the subset of probability measures satisfying*

$$\mathcal{R}_{H^1}(\mu) = \infty \,, \tag{2.24}$$

the following holds. For any $\delta > 0$, there exist probability measures $\mu_0 \in \{\mu \,|\, \mathcal{R}_{H^1}(\mu) = \infty\}$ with the following properties:

– *The right hand side of* (2.12) *diverges at a rate at most* $\exp(ck^\delta)$ *as* $k \to \infty$,

$$\exp\Big[\frac{1}{2k}\log\Big(\int d\mu_0(\phi)\|\phi\|_{H^1}^{2k}\Big)\Big] < Ce^{ck^\delta}. \tag{2.25}$$

– *The initial data defined by* μ_0 *as in* (2.23) *satisfies* $\mathrm{Tr}(|S^{(k,1)}\gamma_0^{(k)}|) < \infty$ *for all* $k \in \mathbb{N}$, *but the associated solution to the cubic focusing GP hierarchy displays instantaneous blowup (see below for the precise definition).*

2.3. **Remarks.** We make the following remarks concerning the case (2.24):

- The precise meaning of instantaneous blowup that we are considering is as follows. Let $A_R := \{\phi \in L^2 | \, \|\phi\|_{L^2} = 1\,,\, \|\phi\|_{H^1} \leq R\}$ for $R > 0$, and denote by $\mathbf{1}_{A_R}$ the corresponding characteristic function. Then, for every $R > 0$, there exists $T = T(R) > 0$ such that the sequence of regularized density matrices

$$\Big(\gamma_R^{(k)}(t) := \int d\mu_0(\phi)\mathbf{1}_{A_R}(\phi)(|S_t(\phi)\rangle\langle S_t(\phi)|)^{\otimes k}\Big)_{k\in\mathbb{N}} \tag{2.26}$$

  is a solution to the focusing cubic GP hierarchy in $L^\infty_{t\in[0,T(R))}\mathfrak{H}^1$. However, in the limit $R \to \infty$,

$$\lim_{R\to\infty} \mathrm{Tr}(|S^{(k,1)}[\gamma_R^{(k)}(t)]|) = \infty \quad \forall t > 0\,, \tag{2.27}$$

  for any $k \in \mathbb{N}$. It is in this sense that we say that $\mathrm{Tr}(|S^{(k,1)}[\gamma^{(k)}(t)]|)$ blows up instantaneously for $t > 0$.

- We note that for local well-posedness to hold, it is necessary that μ_0-almost surely, the blowup time, $\tau(\phi) > \epsilon > 0$, is bounded away from zero. In our analysis of the case (2.24), we will construct measures μ_0 for which $\tau(\phi)$ can be arbitrarily small on the support of μ_0. This is only possible when $\|\phi\|_{H^1}$ can be arbitrarily large on the support of μ_0.

3. Proof of Theorem 2.1

In this section, we apply the quantum de Finetti theorem to prove the existence of scattering states for solutions to the defocusing cubic GP hierarchy in 3 dimensions.

To begin with, we observe that the condition (2.2) implies that $E[\phi] \leq R$ holds μ-almost surely.

Lemma 3.1. *Assume that*

$$\int d\mu(\phi)(E[\phi])^{2k} \leq R^{2k} \tag{3.1}$$

holds for some finite constant $R > 0$, *and all* $k \in \mathbb{N}$. *Then,*

$$\mu\Big(\Big\{\,\phi \in L^2(\mathbb{R}^3)\,\Big|\,\|\phi\|_{L^2} = 1\,,\; E[\phi] > R\Big\}\Big) = 0\,. \tag{3.2}$$

Proof. From Chebyshev's inequality, we have that

$$\begin{aligned} &\mu\Big(\Big\{ \phi \in L^2(\mathbb{R}^3) \Big| \, \|\phi\|_{L^2} = 1 \, , \; E[\phi] > \lambda \Big\} \Big) \\ &\leq \; \frac{1}{\lambda^{2k}} \int d\mu(\phi) (E[\phi])^{2k} \leq \frac{R^{2k}}{\lambda^{2k}} \, , \end{aligned} \tag{3.3}$$

and for $\lambda > R$, the right hand side tends to zero when $k \to \infty$. □

Recalling that $\lambda = 1$, the representation (2.1) immediately yields

$$\begin{aligned} \mathrm{Tr}(|S^{(k,1)}[\gamma_0^{(k)}]|) &= \int d\mu(\phi) \|\phi\|_{H^1}^{2k} \\ &\leq \int d\mu(\phi) (1 + 2E[\phi])^k \leq (1+2R)^k \quad \forall k \in \mathbb{N} \, . \end{aligned} \tag{3.4}$$

This implies that μ-almost surely, $\|\phi\|_{H^1}^2 \leq 1 + 2E[\phi] \leq 1 + 2R$, by the same argument as in Lemma 3.1. Thus, Theorem 1.1 implies that μ-almost surely, there exists a unique solution to the defocusing cubic NLS (1.14) with initial data $\phi(0) = \phi$ which exhibits scattering and asymptotic completeness. For notational convenience further below, we denote $g_\pm(\phi) := \phi_\pm$, such that

$$\lim_{t \to \pm\infty} \|e^{-it\Delta} S_t(\phi) - g_\pm(\phi)\|_{H^1} = 0 \, . \tag{3.5}$$

Then, $g_\pm(\phi) = W_\pm^{-1}(\phi)$.

Using the de Finetti representation of the k-particle marginal

$$\gamma^{(k)} = \int d\mu(\phi) \big(|\phi\rangle\langle\phi| \big)^{\otimes k} \, , \tag{3.6}$$

we let

$$\begin{aligned} \gamma_\pm^{(k)} &:= \int d\mu(\phi) \big(|g_\pm(\phi)\rangle\langle g_\pm(\phi)| \big)^{\otimes k} \\ &= \int d\mu_\pm(\phi) \big(|\phi\rangle\langle\phi| \big)^{\otimes k} \, , \end{aligned} \tag{3.7}$$

where $d\mu_\pm(\phi) = d\mu(W_\pm(\phi))$.

It follows from energy conservation and positivity of the potential energy term $\lambda\|\phi\|_{L^4}^4$ that μ-almost surely,

$$\begin{aligned} \|S_t(\phi)\|_{H^1}^2 &\leq 1 + 2E[\phi] \; \leq \; 1 + 2R \\ \|g_+(\phi)\|_{H^1}^2 &\leq 1 + 2E[\phi] \; \leq \; 1 + 2R \, . \end{aligned} \tag{3.8}$$

For $\phi \in H^1$ satisfying $E[\phi] < R$, we have

$$\begin{aligned} \|e^{-it\Delta} S_t(\phi) - g_\pm(\phi)\|_{H^1} &\leq \|e^{-it\Delta} S_t(\phi)\|_{H^1} + \|g_\pm(\phi)\|_{H^1} \\ &\leq 2(1 + 2E[\phi])^{1/2} < 2(1+2R)^{1/2} \end{aligned} \tag{3.9}$$

uniformly in ϕ, and uniformly in $t \in \mathbb{R}$. Thus, we obtain that

$$\begin{aligned} &\lim_{t \to \pm\infty} \int d\mu(\phi) \|e^{-it\Delta} S_t(\phi) - g_\pm(\phi)\|_{H^1} \\ &= \int d\mu(\phi) \lim_{t \to \pm\infty} \|e^{-it\Delta} S_t(\phi) - g_\pm(\phi)\|_{H^1} \\ &= 0 \, , \end{aligned} \tag{3.10}$$

from the dominated convergence theorem.

We may now prove the existence of scattering states at the level of the GP hierarchy. Using Theorem 2.1 and (3.7), we obtain that

$$\begin{aligned} &\mathrm{Tr}\Big(\Big|S^{(k,1)}\Big[U^{(k)}(-t)\gamma^{(k)}(t)-\gamma_+^{(k)})\Big]\Big|\Big) \\ &= \int d\mu(\phi)\mathrm{Tr}\Big(\Big|S^{(k,1)}\Big[\,\big(\,|U(-t)S_t(\phi)\rangle\langle U(-t)S_t(\phi)|\,\big)^{\otimes k} \\ &\qquad\qquad -\big(|g_+(\phi)\rangle\langle g_+(\phi)|\big)^{\otimes k}\Big)\Big]\Big|\Big) \end{aligned} \tag{3.11}$$

Using the identity

$$A_0^{\otimes k}-A_1^{\otimes k} = \sum_{j=0}^{k-1} A_1^{\otimes j}\otimes(A_0-A_1)\otimes A_0^{\otimes k-1-j} \tag{3.12}$$

with $A_0 := |U(-t)S_t(\phi)\rangle\langle U(-t)S_t(\phi)|$ and $A_1 := |g_+(\phi)\rangle\langle g_+(\phi)|$, and

$$\mathrm{Tr}(|S^{(1,1)}[A_0-A_1]|) \le \|e^{-it\Delta}S_t(\phi)-g_+(\phi)\|_{H^1}\big(\,\|S_t(\phi)\|_{H^1}+\|g_+(\phi)\|_{H^1}\,\big) \tag{3.13}$$

we find

$$\begin{aligned} (3.11) \;&\le\; \sum_{j=0}^{k-1}\int d\mu(\phi)\mathrm{Tr}(|S^{(1,1)}[A_0-A_1]|)(\mathrm{Tr}(|S^{(1,1)}[A_1]|))^j\,\mathrm{Tr}(|S^{(1,1)}[A_0]|)^{k-j-1} \\ &\le\; \int d\mu(\phi)\|e^{-it\Delta}S_t(\phi)-g_+(\phi)\|_{H^1}(\,\|S_t(\phi)\|_{H^1}+\|g_+(\phi)\|_{H^1}\,)^{2k-1} \\ &\le\; \sum_{j=0}^{k-1}\Big(\int d\mu(\phi)\|e^{-it\Delta}S_t(\phi)-g_+(\phi)\|_{H^1}^{2k}\Big)^{\frac{1}{2k}} \\ &\qquad\qquad \Big(\int d\mu(\phi)(\,\|S_t(\phi)\|_{H^1}+\|g_+(\phi)\|_{H^1}\,)^{2k}\Big)^{\frac{2k-1}{2k}}. \end{aligned} \tag{3.14}$$

It follows from (3.4) that μ-almost surely, $E[\phi(t)] = E[\phi] < R$. Together with (3.8), this implies

$$(3.14) \le 2^k\Big(\int d\mu(\phi)\|e^{-it\Delta}S_t(\phi)-g_+(\phi)\|_{H^1}^{2k}\Big)^{\frac{1}{2k}}(1+2R)^{\frac{2k-1}{2k}}. \tag{3.15}$$

The right hand side converges to zero as $t\to\infty$, as a consequence of (3.9) and (3.10).

This concludes the proof of Theorem 2.1. □

4. Proof of Theorem 2.2

4.1. **The case $\mathcal{R}_{H^1}(\mu)<\infty$.** Given $\mathcal{R}_{H^1}(\mu)<R$ for some $R<\infty$, it follows from Lemma 3.1 that μ-almost surely, $\|\phi\|_{H^1}<R$.

The focusing cubic NLS, with flow map $\phi\mapsto S_t(\phi)$, is locally well-posed in $H^1(\mathbb{R}^3)$. In particular, there exist constants $T>0$ and $M<\infty$ such that $\|S_t(\phi)\|_{H^1}<M$ for $t\in[0,T]$ where $T=T(\|\phi\|_{H^1})$ is monotonically decreasing, and where $M=M(\|\phi\|_{H^1})<\infty$ is monotonically increasing in $\|\phi\|_{H^1}$ (more details are given in Section 4.1.1 below). Thus, by monotonicity of T and M with respect to $\|\phi\|_{H^1}$, it follows that μ-almost surely, $\|S_t(\phi)\|_{H^1} < M(R) < \infty$ for $t \in [0, T(R)]$.

Therefore, $\gamma^{(k)}(t)$ as given in (1.23), with $k \in \mathbb{N}$, satisfy

$$\sup_{t\in[0,T(R)]} \mathrm{Tr}(|S^{(k,1)}[\gamma^{(k)}]|) < (M(R))^{2k}\ ,\ \ \forall k \in \mathbb{N}\,, \tag{4.1}$$

and hence, $(\gamma^{(k)})_{k\in\mathbb{N}} \in L^\infty_{t\in[0,T(R)]}\mathfrak{H}^1$. This proves the existence of a solution, and its uniqueness follows from Theorem 1.3.

4.1.1. *Monotonicity of the constants T and M with respect to $\|\phi\|_{H^1}$.* We remark that one can take $T(\|\phi\|_{H^1}) \sim \|\phi\|_{H^1}^{-\beta}$ for some $\beta > 0$ and $M(\|\phi\|_{H^1}) \sim \|\phi\|_{H^1}$. For example, this can be easily obtained from applying the estimate (3.42) in [9] to factorized solutions to the GP hierarchy $\Gamma(t) = ((|S_t(\phi)\rangle\langle S_t(\phi)|)^{\otimes k})_{k\in\mathbb{N}}$ with initial data of the focusing cubic NLS satisfying $\|\phi\|_{H^1} < R$, and for parameters $\xi_1 = \frac{1}{2R}$ and $\xi_2 = \frac{1}{4R}$. In this case, we note that $\|\Gamma(t)\|_{\mathcal{H}^1_{\xi_2}} = \sum_{k\geq 1} \xi_2^k \|S_t(\phi)\|^{2k}_{H^1}$, etc, in the notation of [9].

4.2. **The case $\mathcal{R}_{H^1}(\mu) = \infty$.** We will explicitly construct a family of probability measures on the unit sphere in $L^2(\mathbb{R}^3)$ satisfying

$$\mathcal{R}_{H^1}(\mu) = \infty \tag{4.2}$$

with a prescribed maximum rate of divergence, together with

$$\int d\mu(\phi)\|x\phi\|^2_{L^2} < \infty\,, \tag{4.3}$$

and $\mathrm{Tr}(|S^{(k,1)}\gamma_0^{(k)}|) < \infty$ for all $k \in \mathbb{N}$, such that instantaneous blowup occurs for the corresponding initial data.

In fact, we will be more specific, and construct measures μ such that the sequence $(\gamma^{(k)} = \int d\mu(\phi)\big(|\phi\rangle\langle\phi|\big)^{\otimes k})_{k\in\mathbb{N}}$ belongs to the set

$$\mathfrak{H}^{\alpha,r} := \Big\{\, (\gamma^{(k)})_{k\in\mathbb{N}} \,\Big|\, \mathrm{Tr}(|S^{(k,\alpha)}[\gamma^{(k)}]|) < e^{ck^r} \text{ for some constant } c < \infty \Big\} \tag{4.4}$$

for $r \geq 1$, where evidently, $\mathfrak{H}^\alpha = \mathfrak{H}^{\alpha,1}$.

Instead of an exponential growth of order $\int d\mu(\phi)\|\phi\|^{2k}_{H^1} \leq R^k = O(e^{ck})$, our aim is to admit a growth of order $O(e^{ck^r})$ for some arbitrary $r > 1$. We note that any probability measure μ on $L^2(\mathbb{R}^3)$ having the property that

$$\Big(\int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k}\Big)_{k\in\mathbb{N}} \in \mathfrak{H}^{1,r} \setminus \mathfrak{H}^1 \tag{4.5}$$

satisfies (4.2). The parameter $r > 1$ determines the rate of divergence of (2.12).

To construct a measure μ satisfying (4.3) and (4.5), we may, for simplicity, pick μ to be supported on the unit sphere

$$\mathcal{S} := \{\psi \in L^2(\mathbb{R}^3) |\, \|\psi\|_{L^2} = 1\}\,. \tag{4.6}$$

We consider the dyadic decomposition of $\mathcal{S} = \cup_{j\in\mathbb{N}_0}\mathcal{N}_j$ based on the sets

$$\begin{aligned} \mathcal{N}_j &:= \left\{ \phi \in L^2(\mathbb{R}^3) \,\middle|\, \|\phi\|_{L^2} = 1\,,\, 2^{j-1} < \|\phi\|_{\dot H^1} \le 2^j \right\} \\ \mathcal{N}_0 &:= \left\{ \phi \in L^2(\mathbb{R}^3) \,\middle|\, \|\phi\|_{L^2} = 1\,,\, \|\phi\|_{\dot H^1} \le 1 \right\}, \end{aligned} \tag{4.7}$$

where

$$\|f\|_{\dot H^1} = (\int d\xi |\xi|^2 |\widehat{f}(\xi)|^2)^{1/2}\,.$$

We define

$$d\mu_j(\phi) := d\mu(\phi)\mathbf{1}_{\mathcal{N}_j}(\phi)\,.$$

Our goal is to introduce subsets $\mathcal{M}_j \subset \mathcal{N}_j$, for $j \in \mathbb{N}_0$, such that for initial data $\phi^{(j)} \in \mathcal{M}_j$, the blowup time $\tau(\phi^{(j)})$ for the cubic focusing NLS tends to zero as $j \to \infty$.

For $\phi \in \mathcal{N}_j$, one observes that if $E[\phi] = \frac{1}{2}\|\nabla\phi\|_{L^2}^2 - \frac{1}{4}\|\phi\|_{L^4}^4 < 0$, then

$$\|\phi\|_{L^4} \ge 2^{-\frac{1}{4}} 2^{\frac{j}{2}}\,. \tag{4.8}$$

On the other hand, from the Gagliardo-Nirenberg inequality,

$$\|\phi\|_{L^4} \le C\|\phi\|_{\dot H^1}^{3/4}\|\phi\|_{L^2}^{1/4} \le C 2^{\frac{3j}{4}}\,. \tag{4.9}$$

These are the only restrictions on the size of $\|\phi\|_{L^4}$ on $\mathcal{N}_j$.

Moreover, from the uncertainty principle

$$\|\phi\|_{L^2}^2 \le C\|x\phi\|_{L^2}\|\phi\|_{\dot H^1}\,, \tag{4.10}$$

it follows that for $\phi \in \mathcal{N}_j$,

$$\|x\phi\|_{L^2} > C 2^{-j}\,. \tag{4.11}$$

Thus, we define subsets of $\mathcal{N}_j$ given by

$$\begin{aligned} \mathcal{M}_j &:= \left\{ \phi \in L^2(\mathbb{R}^3) \,\middle|\, \|\phi\|_{L^2} = 1\,, \|x\phi\|_{L^2} < b\,,\, 2^{j-1} < \|\phi\|_{\dot H^1} \le 2^j\,, \|\phi\|_{L^4} > C 2^{\frac{5j}{8}} \right\} \\ \mathcal{M}_0 &:= \left\{ \phi \in L^2(\mathbb{R}^3) \,\middle|\, \|\phi\|_{L^2} = 1\,, \|x\phi\|_{L^2} < b\,,\, \|\phi\|_{\dot H^1} \le 1 \right\}, \end{aligned} \tag{4.12}$$

where $b > 0$ is a fixed constant. These sets are non-empty; an example of a function $f_j \in \mathcal{M}_j$ is given by

$$f_j(x) = 2^{3j/2} g(2^j x)\,, \tag{4.13}$$

where $g(x) = e^{-x^2}$ is the standard Gaussian. We define measures μ_j on $L^2(\mathbb{R}^3)$ satisfying

$$\mu_j(\mathcal{M}_j) = \kappa_r (j^{j^{1/\delta}})^{-j} \tag{4.14}$$

for $r > 1$ and $\delta := r - 1$, where the constant κ_r ensures that $\mu := \sum \mu_j$ is a probability measure on $\mathcal{S}$. For instance, we can think of μ_j as the uniform measure concentrated on $\{e^{i\theta} f_j\}_{\theta\in[0,2\pi)}$, which is invariant under multiplication by a phase.

Then, we let

$$\gamma^{(k)} := \int d\mu(\phi) (|\phi\rangle\langle\phi|)^{\otimes k} \quad \forall k \in \mathbb{N}\,, \tag{4.15}$$

and obtain that

$$\begin{aligned}\mathrm{Tr}(|S^{(k,1)}\gamma^{(k)}|) &= \mathrm{Tr}\Big| S^{(k,1)} \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \Big| \\ &= \sum_j \int d\mu_j(\phi)\|\phi\|_{H^1}^{2k} \\ &\leq C\sum_j (j^{j^{1/\delta}})^{-j} 2^{2jk} \\ &\leq Ce^{ck^r}\,, \end{aligned} \tag{4.16}$$

see Lemma 4.1 below. Thus, $\gamma^{(k)} \in \mathfrak{H}^{1,r}$ for $r > 1$.

On the other hand, $(\gamma^{(k)})_{k\in\mathbb{N}} \notin \mathfrak{H}^{1,1}$. This is because if $(\gamma^{(k)})_{k\in\mathbb{N}} \in \mathfrak{H}^{1,1}$, it follows from Chebyshev's inequality (similar to Lemma 3.1) that

$$\mu\Big(\Big\{\phi \in L^2(\mathbb{R}^3)\Big|\|\phi\|_{H^1} > R\Big\}\Big) = 0\,, \tag{4.17}$$

for some $R < \infty$. But this implies that there are some constants $0 < c < C < \infty$ independent of R, and $J > 0$ such that $c\log R < J < C\log R$ for all $R > 1$ sufficiently large, and $\mu(\mathcal{M}_j) = 0$ for all $j > J$. But then, $\mu_j(\mathcal{M}_j) = 0$ for all $j > J$, which contradicts (4.14).

For $\phi \in \mathcal{M}_j$, we have that

$$\begin{aligned}E[\phi] &= \frac{1}{2}\|\nabla\phi\|_{L^2}^2 - \frac{1}{4}\|\phi\|_{L^4}^4 \\ &< \frac{1}{4}(2^{2j} - 2^{\frac{5}{2}j}) \\ &< -C2^{\frac{5}{2}j} \end{aligned} \tag{4.18}$$

for a constant $C > 0$ independent of j. Therefore, by the blowup criterion of Vlasov-Petrishchev-Talanov [52], Zakharov [54], and Glassey [30], the solution $\phi(t)$ with initial data $\phi(0) = \phi$ blows up in finite time in H^1.

Next, we derive an upper bound T_j on the blowup time for solutions of the focusing cubic NLS with initial data $\phi \in \mathcal{M}_j$. From (2.21), we obtain the quadratic inequality

$$\begin{aligned} 0 &= \|x\phi(0)\|_{L^2}^2 + 2t\|x\phi\|_{L^2}\|\phi\|_{\dot{H}^1} + 8t^2E[\phi] \\ &\leq b^2 + 2tb - 8t^2C2^{\frac{5}{2}j}\,. \end{aligned} \tag{4.19}$$

The positive zero $T_j > 0$ of the quadratic polynomial in t on the lower line provides an upper bound on the blowup time of the solution $\phi(t)$. From (4.19), we get

$$T_j < C2^{-\frac{5j}{2}} \tag{4.20}$$

for a positive constant C independent of j.

Hence, for any $\epsilon > 0$, there exists $J = J(\epsilon) > c|\log\epsilon| > 0$ such that

$$\mathrm{Tr}\Big(\Big|S^{(k,1)}\Big[\sum_{j=0}^{J}\int d\mu_j(\phi)(|S_t(\phi)\rangle\langle S_t(\phi)|)^{\otimes k}\Big]\Big|\Big) \tag{4.21}$$

blows up in a time interval $[0, 2^{-cJ}) \subset [0, \epsilon)$. Letting $\epsilon \to 0$ so that $J \to \infty$, we obtain that $\mathrm{Tr}(|S^{(k,1)}[\gamma^{(k)}(t)]|)$ blows up instantaneously.

This completes the proof of Theorem 2.2. □

Finally, we prove the last step in (4.16).

Lemma 4.1. *Assume that $r > 1$, and let $\delta := r - 1$. Then, for $k \in \mathbb{N}$ sufficently large (depending only on δ),*

$$\sum_j (j^{j^{1/\delta}})^{-j} 2^{2jk} \leq e^{ck^r} \tag{4.22}$$

for a finite constant $c > 0$.

Proof. Clearly,

$$\sum_j (j^{j^{1/\delta}})^{-j} 2^{2jk} = \sum_j \left(\frac{2^{2k}}{j^{j^{1/\delta}}} \right)^j . \tag{4.23}$$

Let $J = J(k) = k^\delta$. Then,

$$\frac{2^{2k}}{j^{j^{1/\delta}}} < \frac{2^{2k}}{k^{\delta k}} < \frac{1}{2}$$

for all $j > J$, if k is large enough (depending only on δ). Therefore,

$$\sum_{j>J} \left(\frac{2^{2k}}{j^{j^{1/\delta}}} \right)^j < \sum_{j>J} \left(\frac{1}{2} \right)^j < 1 , \tag{4.24}$$

for k sufficiently large. On the other hand,

$$\sum_{0 \leq j \leq J} \left(\frac{2^{2k}}{j^{j^{1/\delta}}} \right)^j \leq \sum_{0 \leq j \leq J} 2^{2kj} \leq J 2^{2kJ} = k^\delta 2^{2k^{1+\delta}} \leq e^{ck^r} , \tag{4.25}$$

for a suitable constant $c > 0$, as claimed. □

Acknowledgements. We are grateful to the anonymous referees for their very useful comments. The work of T.C. was supported by NSF grants DMS-1009448 and DMS-1151414 (CAREER). The work of N.P. was supported by NSF grant DMS-1101192. The work of R.S. was supported by NSERC.

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