# A $q$-ANALOGUE OF DE FINETTI'S THEOREM

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ABSTRACT. A $q$-analogue of de Finetti's theorem is obtained in terms of a boundary problem for the $q$-Pascal graph. For $q$ a power of prime this leads to a characterisation of random spaces over the Galois field $\mathbb{F}_q$ that are invariant under the natural action of the infinite group of invertible matrices with coefficients from $\mathbb{F}_q$.

## 1. Introduction

The infinite symmetric group $\mathfrak{S}_\infty$ consists of bijections $\{1,2,\dots\} \to \{1,2,\dots\}$ which move only finitely many integers. The group $\mathfrak{S}_\infty$ acts on the product space $\{0,1\}^\infty$ by permutations of the coordinates. A random element of this space, that is a random infinite binary sequence, is called *exchangeable* if its probability law is invariant under the action of $\mathfrak{S}_\infty$. De Finetti's theorem asserts that every exchangeable sequence can be generated in a unique way by the following two-step procedure: first choose at random the value of parameter $p$ from some probability distribution on the unit interval $[0,1]$, then run an infinite Bernoulli process with probability $p$ for 1's.

One approach to this classical result, as presented in Feller [3, Ch. VII, §4], is based on the following exciting connection with the Hausdorff moment problem. By exchangeability, the law of a random infinite binary sequence is determined by the array $(v_{n,k})$, where $v_{n,k}$ equals the probability of every initial sequence of length $n$ with $k$ 1's. The rule of addition of probabilities yields the backward recursion

$$v_{n,k} = v_{n+1,k} + v_{n+1,k+1},\ 0 \le k \le n,\ n = 0,1,\dots, \tag{1}$$

which readily implies that the array can be derived by iterated differencing of the sequence $(v_{n,0})_{n=0,1,\dots}$. Specifically, setting

$$u_l^{(k)} = v_{l+k,k}, \qquad l = 0,1,\dots, \quad k = 0,1,\dots, \tag{2}$$

and denoting by $\delta$ the difference operator acting on sequences $u = (u_l)_{l=0,1,\dots}$ as

$$(\delta u)_l = u_l - u_{l+1},$$

the recursion (1) can be written as

$$u^{(k)} = \delta u^{(k-1)}, \qquad k = 1,2,\dots. \tag{3}$$

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Since $v_{n,k} \geq 0$, the sequence $u^{(0)}$ must be completely monotone, that is, componentwise

$$\underbrace{\delta \circ \cdots \circ \delta}_{k} u^{(0)} \geq 0, \qquad k = 0, 1, \ldots,$$

but then Hausdorff's theorem implies that there exists a representation

$$v_{n,k} = u^{(k)}_{n-k} = \int_{[0,1]} p^k (1-p)^{n-k} \mu(\mathrm{d}p) \tag{4}$$

with uniquely determined probability measure $\mu$. De Finetti's theorem follows since $v_{n,k} = p^k(1-p)^{n-k}$ for the Bernoulli process with parameter $p$. See [1] for other proofs and extensive survey of generalisations of this result.

The present note is devoted to variations on the $q$-analogue of de Finetti's theorem, which was briefly outlined in Kerov [9] within the framework of the boundary problem for generalised Stirling triangles. The boundary problem for other weighted versions of the Pascal triangle was studied in [4], [6], and for more general graded graphs in [5], [9], [10].

**Definition 1.1.** Given $q > 0$, let us say that a random binary sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots) \in \{0,1\}^\infty$ is *$q$-exchangeable* if its probability law $\mathbb{P}$ is $\mathfrak{S}_\infty$-quasiinvariant with a specific cocycle, which is uniquely determined by the following condition: Denoting by $\mathbb{P}(\varepsilon_1, \ldots, \varepsilon_n)$ the probability of an initial sequence $(\varepsilon_1, \ldots, \varepsilon_n)$, we have for any $i = 1, \ldots, n-1$

$$\mathbb{P}(\varepsilon_1, \ldots, \varepsilon_{i-1}, \varepsilon_{i+1}, \varepsilon_i, \varepsilon_{i+2}, \ldots, \varepsilon_n) = q^{\varepsilon_i - \varepsilon_{i+1}} \mathbb{P}(\varepsilon_1, \ldots, \varepsilon_n).$$

In words: under an elementary transposition of the form $(\ldots, 1, 0, \ldots) \to (\ldots, 0, 1, \ldots)$, probability is multiplied by $q$.

**Theorem 1.2.** *Assume $0 < q < 1$. There is a bijective correspondence $\mathbb{P} \leftrightarrow \mu$ between the probability laws $\mathbb{P}$ of infinite $q$-exchangeable binary sequences and the probability measures $\mu$ on the closed countable set*

$$\Delta_q := \{1, q, q^2, \ldots\} \cup \{0\} \subset [0,1].$$

More precisely, a $q$-exchangeable sequence can be generated in a unique way by first choosing at random a point $x \in \Delta_q$ distributed according to $\mu$ and then running a certain $q$-analogue of the Bernoulli process indexed by $x$. Each law $\mathbb{P}$ is uniquely determined by the infinite triangular array

$$v_{n,k} := \mathbb{P}(\underbrace{1, \ldots, 1}_{k}, \underbrace{0, \ldots, 0}_{n-k}), \qquad 0 \leq k \leq n < \infty, \tag{5}$$

which in turn is given by a $q$-version of formula (4), with $\Delta$ being replaced by $\Delta_q$ (Theorem 3.2). A similar result with switching the roles of 0's and 1's and replacing $q$ by $q^{-1}$ also holds for $q > 1$.

The rest of the paper is organized as follows. In Section 2 we introduce the $q$-Pascal graph and formulate the $q$-exchangeability in terms of certain Markov chains on this graph. In Section 3 we find a characteristic recursion for the numbers (5), which is a $q$-deformation of (1), and we prove the main result, equivalent to Theorem 1.2, using the method of [10]. In Section 4 we discuss three examples: two $q$-analogues of the Bernoulli process and a $q$-analogue of Pólya's urn process. Finally, in Section 5, for $q$ a power of a prime number, we provide an interpretation of the theorem in terms of random subspaces in an infinite-dimensional vector space over $\mathbb{F}_q$.

## 2. The $q$-Pascal graph

For $q > 0$, the $q$-Pascal graph is a *weighted* directed graph $\Gamma(q)$ on the infinite vertex set

$$\Gamma = \{(l, k) : l, k = 0, 1, \dots\}.$$

Each vertex $(l, k)$ has two weighted outgoing edges $(l, k) \to (l+1, k)$ and $(l, k) \to (l, k+1)$ with weights 1 and $q^l$, respectively. The vertex set is divided into levels $\Gamma_n = \{(l, k) : l + k = n\}$, so $\Gamma = \cup_{n\geq 0}\Gamma_n$ with $\Gamma_0$ consisting of the sole root vertex $(0, 0)$. For a path in $\Gamma$ connecting two vertices $(l, k) \in \Gamma_{l+k}$ and $(\lambda, \varkappa) \in \Gamma_{\lambda+\varkappa}$ we define the weight to be the product of weights of edges along the path. For instance, the weight of $(2, 3) \to (2, 4) \to (3, 4) \to (3, 5)$ is $q^5 = q^2 \cdot 1 \cdot q^3$. Clearly, such a path exists if and only if $\lambda \geq l$, $\varkappa \geq k$.

We shall consider certain transient Markov chains $S = (S_n)$, with state-space $\Gamma$, which start at the root $(0, 0)$ and move along the directed edges, so that $S_n \in \Gamma_n$ for every $n = 0, 1, \dots$. Thus, a trajectory of $S$ is an infinite directed path in $\Gamma$ started at the root.

**Definition 2.1.** Adopting the terminology introduced by Vershik and Kerov (see [9]), we say that a Markov chain $S$ on $\Gamma(q)$ is *central* if the following condition is satisfied for each vertex $(n - k, k) \in \Gamma_n$ visited by $S$ with positive probability: given $S_n = (n - k, k)$, the conditional probability that $S$ follows each particular path connecting $(0, 0)$ and $(n-k, k)$ is proportional to the weight of the path.

**Remark 2.2.** If we only require the centrality condition to hold for all $(l, k) \in \Gamma_\nu$ for fixed $\nu$, then we have it satisfied also for all $(l, k)$ with $l + k \leq \nu$. From this it is easy to see that the centrality condition *implies* the Markov property of $S$ in reversed time $n = \dots, 1, 0$, hence also implies the Markov property in forward time $n = 0, 1, \dots$.

In the special case $q = 1$ Definition 2.1 means that in the Pascal graph $\Gamma(1)$ all paths with common endpoints are equally likely.

Recall a bijection between the infinite binary sequences $(\varepsilon_1, \varepsilon_2, \dots)$ and infinite directed paths in $\Gamma$ started at the root $(0, 0)$. Specifically, given a path, the $n$th digit $\varepsilon_n$ is given the value 0 or 1 depending on whether $l$ or $k$ coordinate is increased by 1. Indentifying a path with a sequence $(n - K_n, K_n)$ (where $0 \leq K_n \leq n$), the correspondence can be written as

$$K_n = \sum_{j=1}^{n} \varepsilon_j\,,\ \varepsilon_n = K_n - K_{n-1},\ n = 1, 2, \dots.$$

**Proposition 2.3.** *By virtue of the bijection between $\{0, 1\}^\infty$ and the paths in $\Gamma$, each $q$-exchangeable sequence corresponds to a central Markov chain on $\Gamma(q)$, and vice versa.*

*Proof.* This follows readily from Remark 2.2, Definitions 1.1 and 2.1 and the structure of $\Gamma(q)$. $\square$

We shall use the standard notation

$$[n] := 1 + q + \ldots + q^{n-1},\ [n]! := [1] \cdot [2] \cdots [n],\ \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}$$

for $q$-integers, $q$-factorials and $q$-binomial coefficients, respectively, with the usual convention that $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ for $n < 0$ or $k < 0$. Furthermore, we set

$$(x, q)_k := \prod_{i=0}^{k-1} (1 - xq^i)\ ,\ 1 \le k \le \infty,$$

with the infinite product ($k = \infty$) considered for $0 < q < 1$.

The following lemma justifies the name of the graph by relating it to the $q$-Pascal triangle of $q$-binomial coefficients.

**Lemma 2.4.** *The sum of weights of all directed paths from the root* $(0, 0)$ *to a vertex* $(n - k, k)$*, denoted* $d_{n,k}$*, is given by*

$$d_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix}. \tag{6}$$

*More generally,* $d_{n,k}^{\nu,\varkappa}$*, the sum of weights of all paths connecting two vertices* $(n-k, k)$ *and* $(\nu - \varkappa, \varkappa)$ *in* $\Gamma$ *is given by*

$$d_{n,k}^{\nu,\varkappa} = q^{(\varkappa-k)(n-k)} \begin{bmatrix} \nu - n \\ \varkappa - k \end{bmatrix}.$$

*Proof.* Note that any path from $(0, 0)$ to $(n-k, k)$ has the second component incrementing by 1 on some $k$ edges $(l_i, i-1) \to (l_i, i)$, where $i = 1, 2, \ldots, k$ and $0 \le l_1 \le \cdots \le l_k \le n-k$, thus the sum of weights is equal to

$$d_{n,k} = \sum_{0 \le l_1 \le \cdots \le l_k \le n-k} q^{l_1 + \cdots + l_k}. \tag{7}$$

This array satisfies the recursion

$$d_{n,k} = q^{n-k} d_{n-1,k-1} + d_{n-1,k}\ ,\ 0 < k < n \tag{8}$$

with the boundary conditions $d_{n,0} = d_{n,n} = 1$. On the other hand, it is well known that the array of $q$-binomial coefficients also satisfies this recursion [8], hence by the uniqueness $d_{n,k}$ is the $q$-binomial coefficient. In the like way the sum of weights of paths from $(n-k, k)$ to $(\nu - \varkappa, \varkappa)$ is

$$d_{n,k}^{\nu,\varkappa} = \sum_{n-k \le l_1 \le \cdots \le l_{k'} \le \nu - \varkappa} q^{l_1 + \cdots + l_{k'}}, \quad k' := \varkappa - k.$$

Comparing with (7) we see that this is equal to $q^{(n-k)k'} \begin{bmatrix} \nu - \varkappa \\ k' \end{bmatrix}$. □

**Remark 2.5.** Changing $(l,k)$ to $(k,l)$ yields the *dual* $q$-Pascal graph $\Gamma^*(q)$, which has the same set of vertices and edges as $\Gamma(q)$, but different weights: the edge $(l,k) \to (l,k+1)$ has now weight 1, and the edge $(l,k) \to (l+1,k)$ has weight $q^k$. The sum of weights of paths in $\Gamma^*$ from $(0,0)$ to $(l,k)$ is again (6), which is related to another recursion for $q$-binomial coefficients, $d_{n,k} = d_{n-1,k-1} + q^k d_{n-1,k}$.

Consider the recursion

$$v_{n,k} = v_{n+1,k} + q^{n-k} v_{n+1,k+1}, \text{ with } v_{0,0} = 1, \tag{9}$$

which is dual to (8), and denote by $\mathcal{V}$ the set of nonnegative solutions to (9).

**Proposition 2.6.** *Formula*

$$\mathbb{P}\{S_n = (n-k,k)\} = d_{n,k} v_{n,k}, \qquad (n-k,k) \in \Gamma$$

*establishes a bijective correspondence* $\mathbb{P} \leftrightarrow v$ *between the probability laws of central Markov chains* $S = (S_n)$ *on* $\Gamma(q)$ *and solutions* $v \in \mathcal{V}$ *to recursion* (9).

*Proof.* Let $S$ be a central Markov chain on $\Gamma$ with probability law $\mathbb{P}$. Observe that the property in Definition 2.1 means precisely that the one-step *backward* transition probabilities (that is, transition probabilities in the inverse time) are of the standard form

$$\mathbb{P}\{S_{n-1} = (n-1,k) \mid S_n = (n,k)\} = \frac{d_{n-1,k}}{d_{n,k}} = \frac{[n-k]}{[n]} \tag{10}$$

$$\mathbb{P}\{S_{n-1} = (n-1,k-1) \mid S_n = (n,k)\} = \frac{d_{n-1,k-1} q^{n-k}}{d_{n,k}} = q^{n-k} \frac{[k]}{[n]} \tag{11}$$

for every such $S$.

Introduce the notation

$$\tilde{v}_{n,k} := \mathbb{P}\{S_n = (n-k,k)\}, \qquad (n-k,k) \in \Gamma. \tag{12}$$

Consistency of the distributions of $S_n$'s amounts to the rule of total probability

$$\tilde{v}_{n,k} = \mathbb{P}\{S_n = (n,k) \mid S_{n+1} = (n+1,k)\} \tilde{v}_{n+1,k} + \mathbb{P}\{S_n = (n,k) \mid S_{n+1} = (n+1,k+1)\} \tilde{v}_{n+1,k+1}. \tag{13}$$

Rewriting (13), using (10) and (11), and setting

$$v_{n,k} = d_{n,k}^{-1} \tilde{v}_{n,k} \tag{14}$$

we get (9), which means that $v \in \mathcal{V}$. Thus, we have constructed the correspondence $\mathbb{P} \mapsto v$.

Conversely, start with a solution $v \in \mathcal{V}$ and pass to $\tilde{v} = (\tilde{v}_{n,k})$ according to (14). For each $n$ consider the measure on $\Gamma_n$ with weights $\tilde{v}_{n,0}, \ldots, \tilde{v}_{n,n}$. Since the weight of the root is 1, it follows from (9) by induction in $n$ that these are probability measures. Again by (9), the marginal measures are consistent with the backward transition probabilities, hence determine the probability law of a central Markov chain on $\Gamma(q)$. Thus, we get the inverse correspondence $v \mapsto \mathbb{P}$. □

By virtue of Propositions 2.3 and 2.6, the law of $q$-exchangeable infinite binary sequence is determined by some $v \in \mathcal{V}$, with the entries $v_{n,k}$ having the same meaning as in (5). In the sequel this law will be sometimes denoted $\mathbb{P}_v$.

## 3. The boundary problem

The set $\mathcal{V}$ is a Choquet simplex, meaning a convex set which is compact in the product topology of the space of functions on $\Gamma$ and has the property of uniqueness of the barycentric decomposition of each $v \in \mathcal{V}$ over the set of extreme elements of $\mathcal{V}$ (see, e. g., [7, Proposition 10.21]).

The *boundary problem* for the $q$-Pascal graph amounts to describing extreme nonnegative solutions to the recursion (9). Each extreme solution $v \in \mathcal{V}$ corresponds to ergodic process $(S_n)$ for which the tail sigma-algebra is trivial. In this context, the set of extremes is also known as *the minimal boundary.*

With each array $v \in \mathcal{V}$, $v = (v_{n,k})$, it is convenient to associate another array $\tilde{v} = (\tilde{v}_{n,k})$ related to $v$ via (14). Clearly, the mapping $v \leftrightarrow \tilde{v}$ is an isomorphism of two Choquet simplexes $\mathcal{V}$ and $\widetilde{\mathcal{V}} = \{\tilde{v}\}$. Recall that the meaning of the quantities $\tilde{v}_{n,k}$ is explained in (12).

A common approach to the boundary problem calls for identifying a possibly larger *Martin boundary* (see [10], [6], [4] for applications of the method). To this end, we need to consider multistep backward transition probabilities, which by Lemma 6 are given by a $q$-analogue of the hypergeometric distribution

$$\begin{aligned}\tilde{v}_{n,k}(\nu, \varkappa) &:= \mathbb{P}\{S_n = (n-k, k) \,|\, S_\nu = (\nu - \varkappa, \varkappa)\} \\ &= q^{(\varkappa - k)(n-k)} \begin{bmatrix} \nu - n \\ \varkappa - k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} \Big/ \begin{bmatrix} \nu \\ \varkappa \end{bmatrix}, \quad k = 0, \dots, n, \qquad (15)\end{aligned}$$

and to examine the limiting regimes for $\varkappa = \varkappa(\nu)$ as $\nu \to \infty$, under which the probabilities (15) converge for all fixed $(n-k, k) \in \Gamma$. If the limits exist, the limiting array

$$\tilde{v}_{n,k} := \lim_{(\nu,\varkappa)} \tilde{v}_{n,k}(\nu, \varkappa)$$

belongs necessarily to $\widetilde{\mathcal{V}}$.

Suppose $0 < q < 1$ and introduce polynomials

$$\Phi_{n,k}(x) := q^{-k(n-k)} x^{n-k} (x, q^{-1})_k, \qquad \widetilde{\Phi}_{n,k} = d_{n,k} \Phi_{n,k}\,, \qquad 0 \le k \le n. \qquad (16)$$

Obviously, the degree of $\Phi_{n,k}$ is $n$; we will consider the polynomial as a function on $\Delta_q$. Observe also that $\Phi_{n,k}(x)$ vanishes at points $x = q^{\varkappa}$ with $\varkappa < k$, because of vanishing of $(x, q^{-1})_k$.

**Lemma 3.1.** *Suppose* $0 < q < 1$, *and let in* (15) *the indices* $n$ *and* $k$ *remain fixed, while* $\nu \to \infty$ *and* $\varkappa = \varkappa(\nu)$ *varies in some way with* $\nu$. *Then the limit of* (15) *is* $\widetilde{\Phi}_{n,k}(q^{\varkappa})$ *if* $\varkappa$ *is constant for large enough* $\nu$. *If* $\varkappa \to \infty$ *then the limit is* $\widetilde{\Phi}_{n,k}(0) = \delta_{n,k}$.

*Proof.* Assume first $\varkappa \to \infty$ and show that the limit of (15) is $\delta_{nk}$. Since the quantities $\tilde{v}_{n,k}(\nu, \varkappa)$, where $k = 0, \dots, n$, form a probability distribution, it suffices to check that the limit exists and is equal to 1 for $k = n$. In this case the right–hand side of (15) becomes

$$\prod_{i=1}^{n} \frac{[\varkappa - n + i]}{[\nu - n + i]}.$$

Because $\lim_{m\to\infty}[m] = 1/(1-q)$ for $q < 1$, this indeed converges to 1 provided that $\varkappa \to \infty$.

Now suppose $\varkappa$ is fixed for all large enough $\nu$. The right-hand side of (15) is 0 for $k > \varkappa$. For $k \leq \varkappa$ using $\lim_{m\to\infty}[m-j]!/[m]! = (1-q)^j$ we obtain

$$\begin{bmatrix} \nu - n \\ \varkappa - k \end{bmatrix} \Big/ \begin{bmatrix} \nu \\ \varkappa \end{bmatrix} = \frac{[\nu - n]!}{[\nu]!} \, \frac{[\nu - \varkappa]!}{[\nu - \varkappa - (n-k)]!} \, \frac{[\varkappa]!}{[\varkappa - k]!}$$

$$\to \frac{(1-q)^k [\varkappa]!}{[\varkappa - k]!} = \widetilde{\Phi}_{n,n}(q^{\varkappa}). \quad (17)$$

$\square$

Part (i) of the next theorem appeared in [9, Chapter 1, Section 4, Corollary 6]. Kerov pointed out that the proof could be concluded from the Kerov-Vershik 'ring theorem' (see [5, Section 8.7]), but did not give details.

For $\mu$ a measure, we shall write $\mu(x)$ instead of $\mu(\{x\})$, meaning atomic mass at $x$.

**Theorem 3.2.** *Assume* $0 < q < 1$.

(i) *The formulas*

$$\tilde{v}_{n,k} = \sum_{x \in \Delta_q} \widetilde{\Phi}_{n,k}(x)\mu(x), \qquad v_{n,k} = \sum_{x \in \Delta_q} \Phi_{n,k}(x)\mu(x)$$

*establish a linear homeomorphism between the set* $\widetilde{\mathcal{V}}$ *(respectively,* $\mathcal{V}$*) and the set of all probability measures* $\mu$ *on* $\Delta_q$.

(ii) *Given* $\tilde{v} \in \widetilde{\mathcal{V}}$, *the corresponding measure* $\mu$ *is determined by*

$$\mu(q^{\varkappa}) = \lim_{\nu\to\infty} \tilde{v}_{\nu,\varkappa}, \qquad \varkappa = 0, 1, \dots; \qquad \mu(0) = 1 - \sum_{\varkappa \in \{0,1,\dots\}} \mu(q^{\varkappa}).$$

*Proof.* As in [10], the assertions (i) and (ii) are consequences of the following claims (a), (b) and (c).

(a) For each $\nu = 0, 1, 2, \dots$, the vertex set $\Gamma_\nu$ is embedded into $\Delta_q$ via the map $(\nu, \varkappa) \mapsto q^{\varkappa}$. Observe that, as $\nu \to \infty$, the image of $\Gamma_\nu$ in $\Delta_q$ expands and in the limit exhausts the whole set $\Delta_q$, except point 0, which is a limit point. In this sense, $\Delta_q$ is approximated by the sets $\Gamma_\nu$ as $\nu \to \infty$.

(b) The multistep backward transition probabilities (15) converge to $\widetilde{\Phi}_{n,k}(q^{\varkappa})$, for $0 \le \varkappa \le \infty$, in the regimes described by Lemma 3.1.

(c) The linear span of the functions $\widetilde{\Phi}_{n,k}(x)$, $(n-k,k) \in \Gamma$, is the space of all polynomials, so that it is dense in the Banach space $C(\Delta_q)$. □

Note that part (ii) of the theorem can be rephrased as follows: given $\tilde{v} \in \widetilde{\mathcal{V}}$, consider the probability distribution on $\Gamma_n$ determined by $\tilde{v}_{n,\bullet}$ and take its pushforward under the embedding $\Gamma_\nu \hookrightarrow \Delta_q$. The resulting probability measure on $\Delta_q$ weakly converges to $\mu$ as $n \to \infty$.

**Corollary 3.3.** *For $0 < q < 1$ we have:*

(i) *The extreme elements of $\mathcal{V}$ are parameterised by the points $x \in \Delta_q$ and have the form*

$$v_{n,k} = \Phi_{n,k}(x), \qquad 0 \le k \le n. \tag{18}$$

(ii) *The Martin boundary of the graph $\Gamma(q)$ coincides with its minimal boundary and can be identified with $\Delta_q \subset [0,1]$ via the function $v \mapsto v_{1,0}$.*

*Proof.* All the claims are immediate. We only comment on the fact the parameter $x \in \Delta_q$ is recovered as the value of $v_{1,0}$: this holds because $\Phi_{1,0}(x) = x$. □

Letting $q \to 1$ we have a phase transition: the discrete boundary $\Delta_q$ becomes more and more dense and eventually fills the whole of $[0,1]$ at $q = 1$.

As is seen from (16), the polynomial $\Phi_{n,k}(x)$ can be viewed as a $q$-analogue of the polynomial $x^{n-k}(1-x)^k$, so that (18) is a $q$-analogue of (4). Keep in mind that $x = q^{\varkappa}$ is a counterpart of $1-p$, the probability of $\varepsilon_1 = 0$. The following $q$-analogue of the Hausdorff problem of moments emerges. Introduce a modified difference operator acting on sequences $u = (u_l)_{l=0,1,\dots}$ as

$$(\delta_q u)_l = q^{-l}(u_l - u_{l+1}), \qquad l = 0, 1, \dots .$$

**Corollary 3.4.** *Assume $0 < q < 1$. A real sequence $u = (u_l)_{l=0,1,\dots}$ with $u_0 = 1$ is a moment sequence of a probability measure $\mu$ supported by $\Delta_q \subset [0,1]$ if and only if $u$ is 'q-completely monotone' in the sense that for every $k = 0, 1, \dots$ we have componentwise*

$$\underbrace{\delta_q \circ \cdots \circ \delta_q}_{k} u \ge 0, \qquad k = 0, 1, \dots .$$

*Proof.* Using the notation $v_{l+k,k} = u_l^{(k)}$ as in (2), we see that the recursion (9) is equivalent to $u^{(k)} = \delta_q u^{(k-1)}$, cf. (3). Then we use the fact that $\Phi_{n,0}(x) = x^n$ and repeat in the reverse order the argument of Section 1. □

The case $q > 1$. This case can be readily reduced to the case with parameter $0 < \bar{q} < 1$, where $\bar{q} := q^{-1}$. It is convenient to adopt a more detailed notation $[n]_q$ for the $q$-integers.

**Lemma 3.5.** *For every $q > 0$, $\bar{q} = q^{-1}$, the backward transition probabilities* (10), (11) *for the graph $\Gamma(q)$ and the dual graph $\Gamma^*(\bar{q})$ are the same.*

*Proof.* Indeed, by virtue of (10), (11), this is reduced to the equality

$$\frac{[n-k]_q}{[n]_q} = \bar{q}^k \frac{[n-k]_{\bar{q}}}{[n]_{\bar{q}}}.$$

□

The lemma implies that the boundary problem for $q > 1$ can be treated by passing to $q^{-1} < 1$ and changing $(l, k)$ to $(k, l)$. In terms of the binary encoding of the path, this means switching 0's with 1's.

Kerov [9, Chapter 1, Section 2.2] gives more examples of 'similar' graphs, which have different edge weights but the same backward transition probabilities.

## 4. Examples

A $q$-analogue of the Bernoulli process. Our first example is a description of the extreme $q$-exchangeable infinite binary sequences.

With each infinite binary sequence we associate some *$T$-sequence* $(T_0, T_1, T_2, \dots)$ of nonnegative integers, where $T_j$ is the length of $j$th run of 0's. That is to say, $T_0$ is the number of 0's before the first 1, $T_1$ is the number of 0's between the first and second 1's, $T_2$ is the number of 0's between the second and third 1's, and so on. Clearly, this is a bijection, i.e. a binary sequence can be recovered from its $T$-sequence as

$$(\underbrace{0, \dots, 0}_{T_0}, 1, \underbrace{0, \dots, 0}_{T_1}, 1, \underbrace{0, \dots, 0}_{T_2}, 1, \dots).$$

If $q = 1$, then the Bernoulli process with parameter $p$ has a simple description in terms of the associated random $T$-sequence: all $T_i$ are independent and have the same geometric distribution with parameter $1 - p$.

**Proposition 4.1.** *Assume $0 < q < 1$. For $x \in \Delta_q$, let $v(x) = (v_{n,k}(x))$ be the extreme element of $\mathcal{V}$ corresponding to $x$. Consider $q$-exchangeable infinite binary sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots)$ under the probability law $\mathbb{P}_{v(x)}$ and let $(T_0, T_1, \dots)$ be the associated random $T$-sequence.*

(i) *If $x = q^\varkappa$ with $\varkappa = 1, 2, \dots$ then $T_0, \dots, T_{\varkappa-1}$ are independent, $T_\varkappa \equiv \infty$, and $T_i$ has geometric distribution with parameter $q^{\varkappa-i}$ for $0 \le i \le \varkappa - 1$.*

(ii) *If $x = 1$ then $T_0 \equiv \infty$, which means that with probability one $\varepsilon$ is the sequence $(0, 0, \dots)$ of only 0's.*

(iii) *If $x = 0$ then $T_0 \equiv T_1 \equiv \dots \equiv 0$, which means that with probability one $\varepsilon$ is the sequence $(1, 1, \dots)$ of only 1's.*

*Proof.* Consider the central Markov chain $S = (S_n)$ corresponding to the extreme element $v(q^\varkappa)$. Computing the forward transition probabilities, from (18) and (10), for $0 \le k \le \varkappa$ we have

$$\mathbb{P}\{S_{n+1} = (n+1-k,k) \mid S_n = (n-k,k)\} \\ = \frac{(q^{n+1-k}-1)}{(q^n-1)} \frac{d_{n+1,k}\,\Phi_{n+1,k}(q^\varkappa)}{d_{n,k}\,\Phi_{n,k}(q^\varkappa)} = q^{\varkappa-k}. \quad (19)$$

This implies (i) and (ii). In the limit case $x = 0$ corresponding to $\varkappa \to +\infty$, the above probability equals 0, which entails (iii). □

The analogy with the Bernoulli process is evident from the above description of the binary sequence $\varepsilon(q^\varkappa)$. Moreover, the Bernoulli process appears as a limit. Indeed, fix $p \in (0,1)$ and suppose $\varkappa$ varies with $q$, as $q \uparrow 1$, in such a way that

$$\varkappa \sim \frac{-\log(1-p)}{1-q}.$$

In this limiting regime, $q^{\varkappa-k} \to 1-p$ for every $k$, hence $(T_0, T_1, \dots)$ weakly converges to an infinite sequence of i.i.d. geometric variables with parameter $1-p$, and the random binary sequence $\varepsilon(q^\varkappa)$ converges in distribution to the Bernoulli process with the frequency of 0's equal to $1-p$.

Another $q$-analogue of Bernoulli process. Following [9], another $q$-analogue of Bernoulli process is suggested by the $q$-binomial formula (see [8])

$$(-\theta, q)_n = \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \theta^k.$$

For $\theta \in [0,\infty]$ we define a probability law $\mathbb{P}_{w^\theta}$ for $S = (S_n)$ by setting

$$w^\theta_{n,k} := \frac{\theta^k q^{k(k-1)/2}}{(-\theta, q)_n}, \quad \mathbb{P}_{w^\theta}\{S_n = (n-k,k)\} := d_{n,k} w^\theta_{n,k}, \quad (n,k) \in \Gamma. \qquad (20)$$

Checking (9) is immediate. Computing forward transition probabilities,

$$\mathbb{P}_{w^\theta}\{S_{n+1} = (n+1-k,k) \mid S_n = (n-k,k)\} = 1/(1+\theta q^n),$$

shows that under $\mathbb{P}_{w^\theta}$ the process $S_n = (n - K_n, K_n)$ has independent inhomogeneous increments, with probability $\theta q^{n-1}/(1+\theta q^{n-1})$ for increment $K_n - K_{n-1} = 1$. For $q = 1$ we are back to the ergodic Bernoulli process, but for $0 < q < 1$ the process is not extreme. To obtain the barycentric decomposition of $w^\theta$ over extremes,

$$w^\theta = \sum_{0 \le \varkappa \le \infty} v^\varkappa \mu(q^\varkappa),$$

we can apply Theorem 3.2(ii) to compute from (20)

$$\mu(q^{\varkappa}) = \lim_{n\to\infty} \mathbb{P}_{w^\theta}\{S_n = (n-\varkappa, \varkappa)\} = \frac{1}{(-\theta, q)_\infty} \frac{q^{\varkappa(\varkappa-1)/2}\theta^{\varkappa}}{(1-q)^{\varkappa}[\varkappa]!}.$$

This measure $\mu$ may be viewed as a $q$-analogue of the Poisson distribution.

A $q$-analogue of Pólya's urn process. The conventional Pólya's urn process is described in [3, Section 7.4]. Here we provide its natural deformation.

Fix $a$, $b > 0$ and $0 < q < 1$. Consider the Markov chain $(S_n)$ on $\Gamma$ with the forward transition probabilities from $(n-k, k)$ to $(n+1-k, k)$ and from $(n-k, k)$ to $(n-k, k+1)$ given by

$$\frac{[b+n-k]}{[a+b+n]} \quad \text{and} \quad \frac{[a+k]}{[a+b+n]}\, q^{n-k+b},$$

respectively. Then the distribution at time $n$ is

$$\mathbb{P}\{S_n = (n-k,k)\} = \begin{bmatrix} n \\ k \end{bmatrix} q^{bk} \times \frac{[a][a+1]\cdots[a+k-1][b][b+1]\cdots[b+n-k-1]}{[a+b][a+b+1]\cdots[a+b+n-1]}. \quad (21)$$

Checking consistency (9) is easy. The conventional Pólya's urn process appears in the limit $q \to 1$. The corresponding probability measure $\mu$ is computable from Theorem 3.2(ii) as

$$\lim_{n\to\infty} \mathbb{P}\{S_n = (n-\varkappa, \varkappa)\}$$

For $a = 1$, the limit distribution of the coordinate $\varkappa$ is geometric with parameter $1 - q^b$. For general $a, b$ we obtain a measure on $\Delta_q$

$$\mu(q^{\varkappa}) = \frac{(q^a, q)_{\varkappa}(q^b, q)_\infty}{(q, q)_{\varkappa}(q^{a+b}; q)_\infty}\, q^{\varkappa b}, \qquad q^{\varkappa} \in \Delta_q,$$

which may be viewed as a $q$-analogue of the beta distribution on $[0, 1]$.

## 5. Grassmannians over a finite field

For $q$ a power of a prime number, let $\mathbb{F}_q$ be the Galois field with $q$ elements. Define $V_n$ to be the $n$-dimensional space of sequences $(\xi_1, \xi_2, \ldots)$ with entries from $\mathbb{F}_q$, which satisfy $\xi_i = 0$ for $i > n$. The spaces $\{0\} = V_0 \subset V_1 \subset V_2 \subset \ldots$ comprise a complete flag, and the union $V_\infty := \cup_{n\ge 0} V_n$ is a countable, infinite-dimensional space over $\mathbb{F}_q$.

By the *Grassmannian* $\mathrm{Gr}(V_\infty)$ we mean the set of all vector subspaces $X \subseteq V_\infty$. Likewise, for $n \ge 0$ let $\mathrm{Gr}(V_n)$ be the set of all vector subspaces in $V_n$, with $\mathrm{Gr}(V_0)$ being a singleton. Consider the projection $\pi_{n+1,n} : \mathrm{Gr}(V_{n+1}) \to \mathrm{Gr}(V_n)$ which sends a subspace of $V_{n+1}$ to its intersection with $V_n$.

**Lemma 5.1.** *There is a canonical bijection $X \leftrightarrow (X_n)$ between the Grassmannian* $\mathrm{Gr}(V_\infty)$ *and the set of sequences* $(X_n \in \mathrm{Gr}(V_n),\ n \ge 0)$ *satisfying the consistency condition* $X_n = \pi_{n+1,n}(X_{n+1})$ *for each* $n$.

*Proof.* Indeed, the mapping $X \mapsto (X_n)$ is given by setting $X_n = X \cap V_n$ for each $n$, while the mapping $(X_n) \mapsto X$ is defined by $X = \cup X_n$. □

The lemma shows that $\mathrm{Gr}(V_\infty)$ can be identified with a projective limit of the finite sets $\mathrm{Gr}(V_n)$, the projections being the maps $\pi_{n+1,n}$. Using this identification we endow $\mathrm{Gr}(V_\infty)$ with the corresponding topology, in which $\mathrm{Gr}(V_\infty)$ becomes a totally disconnected compact space. For $X \in \mathrm{Gr}(V_\infty)$, a fundamental system of its neighborhoods is comprised of the sets of the form $\{X' \in \mathrm{Gr}(V_\infty) :\ X'_n = X_n\}$, where $n = 1, 2, \dots$ .

Let $\mathscr{G}_n = GL(n, \mathbb{F}_q)$ be the group of invertible linear transformations of the space $V_n$, realised as the group of transformations of $V_\infty$ which may only change the first $n$ coordinates. We have then $\{e\} = \mathscr{G}_0 \subset \mathscr{G}_1 \subset \mathscr{G}_2 \subset \dots$ and we define $\mathscr{G}_\infty := \cup \mathscr{G}_n$. The countable group $\mathscr{G}_\infty$ consists of infinite invertible matrices $(g_{ij})$, such that $g_{ij} = \delta_{ij}$ for large enough $i + j$. The group $\mathscr{G}_\infty$ acts on $V_\infty$ hence also acts on $\mathrm{Gr}(V_\infty)$.

A probability distribution on $\mathrm{Gr}(V_\infty)$ defines a random subspace of $V_\infty$. We look at random subspaces of $V_\infty$ whose distribution is invariant under the action of $\mathscr{G}_\infty$. Observe that the action of $\mathscr{G}_n$ splits $\mathrm{Gr}(V_n)$ into orbits

$$G(n,k) = \{X \in \mathrm{Gr}(V_n), \dim X = k\},\ 0 \le k \le n,$$

where $\#G(n,k) = d_{n,k}$ is the number of $k$-dimensional subspaces of $V_n$. Therefore, a probability distribution on $\mathrm{Gr}(V_\infty)$ is $\mathscr{G}_\infty$-invariant if and only if the conditional distribution on each $G(n,k)$ is uniform.

It must be clear that this setting of '$q$-exchangeability' of linear spaces is analogous to the framework of de Finetti's theorem: exchangeability of a random binary sequence means that the conditional measure is uniform on sequences of length $n$ with $k$ 1's. See [1], [2] for more on symmetries and sufficiency.

**Lemma 5.2.** *Formula*

$$\tilde v_{n,k} = P\{X \in \mathrm{Gr}(V_\infty) : X \cap V_n \in G(n,k)\}, \quad (n,k) \in \Gamma$$

*establishes a linear homeomorphism between* $\widetilde{\mathcal{V}}$ *and* $\mathscr{G}_\infty$*-invariant probability measures on the Grassmannian* $\mathrm{Gr}(V_\infty)$.

*Proof.* We first spell out more carefully the remark before the lemma. Consider projections

$$\pi_{\infty,n} : \mathrm{Gr}(V_\infty) \to \mathrm{Gr}(V_n), \qquad X \mapsto X \cap V_n, \quad X \in \mathrm{Gr}(V_\infty), \quad n = 1, 2, \dots .$$

If $P$ is a Borel probability measure on the space $\mathrm{Gr}(V_\infty)$, then, for any $n$, the pushforward $P_n := \pi_{\infty,n}(P)$ is a probability measure on $\mathrm{Gr}(V_n)$, and the measures $P_n$ are consistent with respect to the projections $\pi_{n+1,n}$, that is,

$$P_n = \pi_{n+1,n}(P_{n+1}), \qquad n = 0, 1, 2, \dots .$$

Conversely, if a sequence $(P_n)$ of probability measures is consistent, then it determines a probability measure $P$ on $\mathrm{Gr}(V_\infty)$. Moreover, $P$ is $\mathscr{G}_\infty$-invariant if and only if each $P_n$ is $\mathscr{G}_n$-invariant. Next, observe that if $P_n$ is a $\mathscr{G}_n$-invariant probability measure, then it assigns the same weight to each $k$-dimensional space $X_n \in G(n,k)$; let us denote this weight by $v_{n,k}$.

Fix $X_n \in G(n,k)$. We claim that there are precisely $q^{n-k}+1$ subspaces $X_{n+1} \in \mathrm{Gr}(V_{n+1})$ such that $X_{n+1} \cap V_n = X_n$: one subspace from $G(n+1,k)$ and $q^{n-k}$ subspaces from $G(n+1,k+1)$. Indeed, $\dim X_{n+1}$ equals either $k$ or $k+1$. In the former case $X_{n+1} = X_n$, while in the latter case $X_{n+1}$ is spanned by $X_n$ and a nonzero vector from $V_{n+1} \setminus V_n$. Such a vector is defined uniquely up to a scalar multiple and addition of an arbitrary vector from $X_n$. Therefore, the number of options is equal to the number of lines in $V_{n+1}/X_n$ not contained in $V_n/X_n$, which equals

$$\frac{q^{n+1-k}-1}{q-1} - \frac{q^{n-k}-1}{q-1} = q^{n-k}.$$

Now, let $P$ be a $\mathscr{G}_\infty$-invariant probability measure on $\mathrm{Gr}(V_\infty)$, with projections $(P_n)$ specified by the corresponding array of weights $v = (v_{n,k})$. Then the relations $P_n = \pi_{n+1,n}(P_{n+1})$ together with the dimension computation imply that $v$ satisfies (9).

Conversely, given $v \in \mathcal{V}$, we can construct a sequence $(P_n)$ of measures such that $P_n$ lives on $\mathrm{Gr}(V_n)$, is invariant under $\mathscr{G}_n$ and agrees with $P_{n+1}$ under $\pi_{n+1,n}$. Since $P_0$, which lives on a singleton, is obviously a probability measure, we obtain by induction that all $P_n$ are probability measures. Taking their projective limit we get a $\mathscr{G}_\infty$-invariant probability measure $P$ on $\mathrm{Gr}(V_\infty)$. $\square$

Rephrasing Theorem 3.2 we have from the lemma

**Corollary 5.3.** *The ergodic $\mathscr{G}_\infty$-invariant probability measures on $\mathrm{Gr}(V_\infty)$ are parameterised by $\varkappa \in \{0,1,\dots,\infty\}$. For $\varkappa = 0$ the measure is the Dirac mass at $V_\infty$, for $\varkappa = \infty$ it is the Dirac mass at $V_0$, and for $0 < \varkappa < \infty$ the measure is supported by the set of subspaces of $V_\infty$ of codimension $\varkappa$.*

The following random algorithm describes explicitly the dynamics of the growing space $X_n \in \mathrm{Gr}(V_n)$ as $n$ varies, under the ergodic measure with parameter $\varkappa$. Recall the notation $\bar q = q^{-1}$. Start with $X_0 = V_0$. With probability $\bar q^{\varkappa}$ choose $X_1 = V_1$, and with probability $1-\bar q^{\varkappa}$ choose $X_1 = X_0$. Suppose $X_n \subseteq V_n$ has been constructed and has dimension $n-k$ with $k \le \varkappa$. Then let $X_{n+1} = X_n$ with probability $1-\bar q^{\varkappa-k}$, and with probability $\bar q^{\varkappa-k}$ choose uniformly at random a nonzero vector $\xi \in V_{n+1} \setminus V_n$ and let $X_{n+1}$ be the linear span of $X_n$ and $\xi$.

Duality. We finish with a dual version of our construction. Let $V^\infty$ denote the set of all sequences $\eta = (\eta_1, \eta_2, \dots)$ with entries from $\mathbb{F}_q$. This is again a vector space over $\mathbb{F}_q$, strictly larger than $V_\infty$ since we do not require $\eta$ to have finitely many nonzero entries. That is to say, $V^\infty$ is just the infinite product space $(\mathbb{F}_q)^\infty$, which we endow with the product topology. Let $\mathrm{Gr}(V^\infty)$ denote the set of all *closed* subspaces $Y \subseteq V^\infty$. A dual version of Lemma 5.1 says that such subspaces $Y$ are in a bijective correspondence with the sequences $(Y_n \in \mathrm{Gr}(V_n), n \geq 0)$ such that $Y_n = \pi'_{n+1,n}(Y_{n+1})$, where $\pi'_{n+1,n}$ is induced by the projection map $V_{n+1} \to V_n$ which sets the $(n+1)$th coordinate of a vector $\xi \in V_{n+1}$ equal to 0. The branching of $G(n,k)$'s under these projections corresponds to the dual $q$-Pascal graph.

**Lemma 5.4.** *The operation of passing to the orthogonal complement with respect to the bilinear form*

$$\langle \xi, \eta \rangle := \sum_{i=1}^{\infty} \xi_i \eta_i, \qquad \xi \in V_\infty, \quad \eta \in V^\infty,$$

*is a bijection* $\mathrm{Gr}(V_\infty) \leftrightarrow \mathrm{Gr}(V^\infty)$.

*Proof.* First of all, note that the bilinear form is well defined, because the coordinates $\xi_i$ of $\xi \in V_\infty$ vanish for $i$ large enough. This form determines a bilinear pairing $V_\infty \times V^\infty \to \mathbb{F}_q$. We claim that it brings the spaces $V_\infty$ and $V^\infty$ into duality, where $V^\infty$ is viewed as a vector space with nontrivial topology, and the topology on $V_\infty$ is discrete.

Indeed, it is evident that the pairing is nondegenerate and that any linear functional on $V_\infty$ is given by a vector of $V^\infty$. A minor reflection also shows that, conversely, any *continuous* linear functional on $V^\infty$ is given by a vector from $V_\infty$. Thus, the spaces $V_\infty$ and $V^\infty$ are indeed dual to one another. They are also dual as commutative locally compact topological groups: one is discrete and the other is compact.

Using the duality, it is readily checked that if $X$ is an arbitrary subspace in $V_\infty$, then its orthogonal complement $X^\perp$ is a closed subspace in $V^\infty$, whose orthogonal complement $(X^\perp)^\perp$ coincides with $X$. Likewise, starting with a closed subspace $Y \subseteq V^\infty$, we have $Y^\perp \subseteq V_\infty$ and $(Y^\perp)^\perp = Y$. Thus, the operation of taking the orthogonal complement is a bijection. □

The group $\mathscr{G}_\infty$ acts on both $V_\infty$ and $V^\infty$ and preserves the pairing between these vector spaces. Under the identification $\mathrm{Gr}(V^\infty) = \mathrm{Gr}(V_\infty)$, the group $\mathscr{G}_\infty$ acts by homeomorphisms on this compact space. In the dual picture, the ergodic measures with $\varkappa < \infty$ live on the set of $\varkappa$-dimensional subspaces of $V^\infty$ . The case $\varkappa = \infty$ corresponds then to the zero subspace in $V_\infty$ (or the full space $V^\infty$). There is a simple explanation why we have to fix codimension in the $V_\infty$-picture and dimension in the $V^\infty$-picture, and not vice versa. Namely, the subspaces in $V_\infty$ of fixed nonzero finite dimension form a countable set, which is a single $\mathscr{G}_\infty$-orbit, and such a $\mathscr{G}_\infty$-space cannot carry a finite invariant measure.

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