A NONCOMMUTATIVE EXTENDED DE FINETTI THEOREM

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Abstract. The extended de Finetti theorem characterizes exchangeable infinite random sequences as conditionally i.i.d. and shows that the apparently weaker distributional symmetry of spreadability is equivalent to exchangeability. Our main result is a noncommutative version of this theorem.

In contrast to the classical result of Ryll-Nadzewski [RN57], exchangeability turns out to be stronger than spreadability for infinite noncommutative random sequences. Out of our investigations emerges noncommutative conditional independence in terms of a von Neumann algebraic structure closely related to Popa's notion of commuting squares [Pop83b] and Kümmerer's generalized Bernoulli shifts [Küm88b]. Our main result is applicable to classical probability, quantum probability, in particular free probability [VDN92], braid group representations and Jones subfactors [GHJ89, GK08].

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Introduction and main result

The characterization of random objects with distributional symmetries is of major interest in modern probability theory and Kallenberg's recent monograph [Kal05] provides an impressive account on this subject. Already in the early 1930s, de Finetti showed that infinite exchangeable random sequences are conditionally i.i.d. or, more intuitively formulated, mixtures of i.i.d. random variables [Fin31, CR96]. An early version of his celebrated characterization is that for every infinite sequence of exchangeable $\{0,1\}$-valued random variables $X \equiv$

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$(X_1, X_2, X_3, \ldots)$, there exists a probability measure ν on $[0,1]$ such that the law $\mathcal{L}(X)$ is given by

$$\mathcal{L}(X) = \int_{[0,1]} m(p) d\nu(p).$$

Here $m(p)$ denotes the infinite product of the measure with Bernoulli distribution $(p, 1-p)$. An extension of this result from the set $\{0,1\}$ to any compact Hausdorff space Ω goes back to Hewitt and Savage [HS55] and soon after it was realized by Ryll-Nadzewski [RN57] that the apparently weaker distributional symmetry of spreadability is equivalent to exchangeability for infinite random sequences. A further extension to standard Borel spaces is provided by Aldous in his monograph on exchangeability [Ald85]. Let us mention that 'spreadability' is also known as 'contractability' in the probability theory and shares common ground with 'subsymmetric sequences' in Banach space theory.

Our main result is a noncommutative version of the following extended de Finetti theorem. We have adapted its formulation in [Kal05, Theorem 1.1] to the purposes of this paper:

Theorem 0.1. *Let $X \equiv (X_n)_{n\in\mathbb{N}} \colon (\Omega, \Sigma, \mu) \to (\Omega_0, \Sigma_0)$ be a sequence of random variables, where (Ω, Σ) and (Ω_0, Σ_0) are standard Borel spaces and μ is a probability measure. Then the following conditions are equivalent:*

(a) *X is exchangeable;*
(b) *X is spreadable;*
(c) *X is conditionally i.i.d.*

Here the conditioning is with respect to the tail σ-field of the random sequence X. Three different proofs of this result can be found in [Kal05] and it is worthwhile to point out that the two implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) are fairly clear; the main work rests on proving the implication (b) $\Rightarrow$ (c).

An early noncommutative version of de Finetti's theorem was given by Størmer for exchangeable states on the infinite tensor product of C*-algebras [Stø69]. His pioneering work stimulated further results in quantum statistical physics and quantum probability, with focus on bosonic systems [HM76, Hud81, FLV88]. A quite general noncommutative analogue of de Finetti's theorem is obtained by Accardi and Lu in a C*-algebraic setting, where only the tail algebra (generated by the exchangeable infinite noncommutative random sequence) is required to be commutative [AL93]. Quite recently, inspired by Good's formula and Speicher's free cumulants [Spe98], a combinatorial approach by Lehner unifies cumulant techniques in a *-algebraic setting of exchangeability systems [Leh04, Leh03, Leh05, Leh06]. He shows that exchangeability entails properties of cumulants, as they are known in classical probability to be characterizing for (conditional) independence. Presently, no results on noncommutative versions of de Finetti's theorem seem to be available in the literature beyond the case of commutative tail algebras and Lehner's combinatorial results for exchangeability systems; and no results at all are present in the noncommutative realm for the extended de Finetti theorem, Theorem 0.1.

Our framework towards a noncommutative version of the extended de Finetti theorem needs to be capable to efficiently handle tail events. This suggests to deal right from the beginning with W*-algebraic probability spaces. We will work with noncommutative probability spaces $(\mathcal{M}, \psi)$ which consist of a von Neumann algebra $\mathcal{M}$ (with separable predual) and a faithful normal state ψ on $\mathcal{M}$. A noncommutative random variable ι from $(\mathcal{A}_0, \varphi_0)$ to $(\mathcal{M}, \psi)$ is given by an injective *-homomorphism $\iota\colon \mathcal{A}_0 \to \mathcal{M}$ such that $\varphi_0 = \psi \circ \iota$. Furthermore, we require that the ψ-preserving conditional expectation from $\mathcal{M}$ onto $\iota(\mathcal{A}_0)$ exists (see Section 1 for further details).

Here we will constrain our investigations to a random sequence $\mathscr{I}$, given by an infinite sequence of *identically distributed* random variables $\iota \equiv (\iota_n)_{n\in\mathbb{N}_0}$ from $(\mathcal{A}_0, \varphi_0)$ to $(\mathcal{M}, \psi)$. This simplification improves the transparency of our approach, since it allows us to realize ι as *injective* mappings from the *single* probability space $(\mathcal{A}_0, \varphi_0)$. A treatment beyond identically distributed random variables is possible and of course of interest; it would start with a probability space $(\mathcal{M}, \psi)$ and a sequence of (not necessarily injective) normal *-homomorphisms from a von Neumann algebra $\mathcal{A}$ into $\mathcal{M}$. Since the distributional symmetries considered herein will lead anyway to stationarity (which implies identical distributions), we omit this primary technical generalization.

We recall that $\mathcal{M}$ is of the form $L^\infty(\Omega, \Sigma, \mu)$ for some standard Borel space (Ω, Σ, μ) as soon as $\mathcal{M}$ is commutative; and then one has $\psi = \int_\Omega \cdot \, d\mu$. In this case a random variable $X\colon (\Omega, \Sigma, \mu) \to (\Omega_0, \Sigma_0)$ reappears as an injective *-homomorphism $\iota\colon L^\infty(\Omega_0, \Sigma_0, \mu_X) \to L^\infty(\Omega, \Sigma, \mu)$ with $\iota(f) := f \circ X$ (the measure μ_X is the distribution of X). Given a sequence of random variables $(X_n)_{n\in\mathbb{N}_0}$, the constraint of identically distributed X_n's ensures that we can identify all image measures μ_{X_n} with the single measure μ_{X_0}. Note that this approach is free of any conditions on the existence of moments of the X_n's.

Throughout we will work with a noncommutative notion of conditional independence which, by our main result, can actually seen to emerge out of the transfer of the extended de Finetti theorem to noncommutative probability. It encompasses Popa's notion of 'commuting squares' in subfactor theory [Pop83b, GHJ89] as well as Voiculescu's freeness with amalgamation [VDN92], aside of tensor independence and many other examples coming from generalized Gaussian random variables [BKS97, GM02].

Consider the random sequence $\mathscr{I}$ which generates the von Neumann subalgebras

$$\mathcal{M}_I := \bigvee_{i\in I} \iota_i(\mathcal{A}_0)$$

for subsets $I \subset \mathbb{N}_0$ and the tail algebra

$$\mathcal{M}^{\mathrm{tail}} := \bigcap_{n\in\mathbb{N}_0} \bigvee_{k\geq n} \iota_k(\mathcal{A}_0).$$

Let $E_{\mathcal{M}^{\mathrm{tail}}}$ denote the ψ-preserving conditional expectation from $\mathcal{M}$ onto $\mathcal{M}^{\mathrm{tail}}$. Then we say that $\mathscr{I}$ is *full* $\mathcal{M}^{\mathrm{tail}}$*-independent* if

$$E_{\mathcal{M}^{\mathrm{tail}}}(xy) = E_{\mathcal{M}^{\mathrm{tail}}}(x)E_{\mathcal{M}^{\mathrm{tail}}}(y)$$

for all $x \in \mathcal{M}^{\mathrm{tail}} \vee \mathcal{M}_I$ and $y \in \mathcal{M}^{\mathrm{tail}} \vee \mathcal{M}_J$ with $I \cap J = \emptyset$. We will also meet a weaker notion of independence, called *order* $\mathcal{M}^{\mathrm{tail}}$*-independence*, which requires the (ordered) sets I and J to satisfy $I < J$ or $I > J$, instead of disjointness.

These two notions of conditional independence do *not* require $\mathcal{M}^{\mathrm{tail}} \subset \mathcal{M}_I$ and allow a noncommutative dual formulation of random measures as they are necessary in the context of de Finetti's theorem. Interesting on its own, the paradigm of an infinite sequence X of exchangeable $\{0,1\}$-valued random variables clearly illustrates that, in its algebraic reformulation, stipulating the inclusion $\mathcal{M}^{\text{tail}} \subset \mathcal{M}_I$ implies the triviality $\mathcal{M}^{\text{tail}} \simeq \mathbb{C}$ and thus forces X to be i.i.d. Thus it is crucial to allow $\mathcal{M}^{\text{tail}} \not\subset \mathcal{M}_I$ if one is interested in transferring results on distributional symmetries to a noncommutative setting.

In order to state our main result, we informally introduce the relevant distributional symmetries. Given the two random sequences $\mathscr{I}$ and $\tilde{\mathscr{I}}$ with random variables ι resp. $\tilde{\iota}$, both from $(\mathcal{A}_0, \varphi_0)$ to $(\mathcal{M}, \psi)$, we write

$$(\iota_0, \iota_1, \iota_2, \ldots) \stackrel{\text{distr}}{=} (\tilde{\iota}_0, \tilde{\iota}_1, \tilde{\iota}_2, \ldots)$$

if $\mathscr{I}$ and $\tilde{\mathscr{I}}$ have the same distribution:

$$\psi\big(\iota_{\mathbf{i}(1)}(a_1)\iota_{\mathbf{i}(2)}(a_2)\cdots\iota_{\mathbf{i}(n)}(a_n)\big) = \psi\big(\tilde{\iota}_{\mathbf{i}(1)}(a_1)\tilde{\iota}_{\mathbf{i}(2)}(a_2)\cdots\tilde{\iota}_{\mathbf{i}(n)}(a_n)\big)$$

for all n-tuples $\mathbf{i}\colon \{1,2,\ldots,n\} \to \mathbb{N}_0$, $(a_1,\ldots,a_n) \in \mathcal{A}_0^n$ and $n \in \mathbb{N}$. Now a random sequence $\mathscr{I}$ is said to be *exchangeable* if its distribution is invariant under permutations:

$$(\iota_0, \iota_1, \iota_2, \ldots) \stackrel{\text{distr}}{=} (\iota_{\pi(0)}, \iota_{\pi(1)}, \iota_{\pi(2)}, \ldots)$$

for any finite permutation $\pi \in \mathbb{S}_\infty$ of $\mathbb{N}_0$. We say that random sequence $\mathscr{I}$ is *spreadable* if every subsequence has the same distribution:

$$(\iota_0, \iota_1, \iota_2, \ldots) \stackrel{\text{distr}}{=} (\iota_{n_0}, \iota_{n_1}, \iota_{n_2}, \ldots)$$

for any subsequence $(n_0, n_1, n_2, \ldots)$ of $(0,1,2,\ldots)$. Finally, $\mathscr{I}$ is *stationary* if the distribution is shift-invariant:

$$(\iota_0, \iota_1, \iota_2, \ldots) \stackrel{\text{distr}}{=} (\iota_k, \iota_{k+1}, \iota_{k+2}, \ldots)$$

for all $k \in \mathbb{N}$.

We are ready to formulate our main result, a noncommutative dual version of Theorem 0.1.

Theorem 0.2. *Let $\mathscr{I}$ be a random sequence with (identically distributed) random variables*

$$\iota \equiv (\iota_i)_{i\in\mathbb{N}_0}\colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{M}, \psi)$$

and consider the following conditions:

(a) *$\mathscr{I}$ is exchangeable;*
(b) *$\mathscr{I}$ is spreadable;*
(c) *$\mathscr{I}$ is stationary and full $\mathcal{M}^{\text{tail}}$-independent;*
(d) *$\mathscr{I}$ is full $\mathcal{M}^{\text{tail}}$-independent;*
(c_o) *$\mathscr{I}$ is stationary and order $\mathcal{M}^{\text{tail}}$-independent;*
(d_o) *$\mathscr{I}$ is order $\mathcal{M}^{\text{tail}}$-independent.*

Then we have the implications:

$$\begin{array}{ccccccc} (a) & \Rightarrow & (b) & \Rightarrow & (c) & \Rightarrow & (d) \\ & & & & \Downarrow & & \Downarrow \\ & & & & (c_o) & \Rightarrow & (d_o) \end{array}$$

Moreover, there are examples such that (a) $\not\Leftarrow$ (b) $\not\Leftarrow$ (c) $\not\Leftarrow$ (d) and (c_o) $\not\Leftarrow$ (d_o).

Similar to the classical case, the hard part of the proof is that spreadability implies conditional *full* independence. This is done by means from noncommutative ergodic theory.

One might object that a noncommutative version of the extended de Finetti theorem should provide an equivalence of these conditions. But our investigations show that such a common folklore understanding would be conceptually misleading in the noncommutative world. The crucial implications from distributional symmetries to conditional (full/order) independence are still valid. All listed converse implications fail due to deep structural reasons, and the others are presently open in the generality of our setting.

The failure of the implication '(b) $\Rightarrow$ (a)' relies on the fact that, in the noncommutative realm, spreadability of infinite random sequences goes beyond the representation theory of the symmetric group. As developed in [GK08], braid group representations with infinitely many generators lead to *braidability*, a new symmetry intermediate to exchangeability and spreadability. This braidability extends exchangeability and provides a rich source of spreadable noncommutative random sequences such that the reverse implication '(b) $\Rightarrow$ (a)' fails. Some of these 'counter-examples' are known in subfactor theory as vertex models from quantum statistical physics. The inequivalence of exchangeability and spreadability is a familiar phenomena for random arrays [Kal05]. Since this phenomena already occurs for infinite sequences in the noncommutative setting, it provides another facet of the common folklore result that $(d+1)$-dimensional classical models correspond to d-dimensional quantum models [EK98].

Examples for the failure of the implication '(c) $\Rightarrow$ (b)' are also available in the context of braid group representations. It is shown in [GK08] that an appropriate cocycle perturbation of the unilateral shift of a stationary random sequence may obstruct spreadability without effecting the structure of conditional full independence. Again, related 'counter-examples' arise in the most natural manner. For example, the symbolic shift on the Artin generators of the braid group $\mathbb{B}_\infty$ induces an endomorphism of the braid group von Neumann algebra $L(\mathbb{B}_\infty)$ such that, when acting on the subalgebra $L(\mathbb{B}_2)$, the resulting stationary random sequence exhibits the failure of '(c) $\Rightarrow$ (b)'. Such phenomena are impossible in the classical case by Theorem 0.1.

Finally, one can not expect in the noncommutative realm that i.i.d. random sequences are automatically stationary. The failure of the implication '(d) $\Rightarrow$ (c)', and thus of '($\mathrm{d_o}$) $\Rightarrow$ ($\mathrm{c_o}$)', is closely related to the fact that our notion of noncommutative conditional independence is more general than (conditioned versions of) tensor independence or free independence. The latter two notions of independence enjoy universality properties [Spe97, BGS02, NS06] which immediately entail stationarity of an i.i.d. random sequence. In particular, they are rigid with respect to certain 'local perturbations' of noncommutative random sequences. But we will see that, starting with a stationary (conditionally full/order) independent random sequence, our more general notion of independence is non-rigid with respect to such 'local perturbations'. Related examples arise again in the context of braid group representations or noncommutative Gaussian random variables. Thus stationarity plays are more distinguished role in the quantum setting and cannot simply be deduced from independence properties as it is the case for classical probability or Voiculescu's free probability.

A closer look at Theorem 0.2 reveals that it is 'dual' to the usual formulations of de Finetti's theorem. In terms of quantum physics, our theorem is formulated in the Heisenberg picture, whereas the usual formulations use the Schrödinger picture.

Or equivalently phrased: our result is on the level of the von Neumann algebra, whereas the latter identify the geometry of exchangeable states in the predual of the von Neumann algebra. Using the theory of noncommutative L^1-spaces it would be of interest to examine in detail the geometry of exchangeable, spreadable or 'conditionally independent' subspaces.

We summarize the content of this paper.

Section 1 introduces our setting of noncommutative probability spaces, random sequences and distributional symmetries. It closes with the proof of some of the elementary implications of Theorem 0.2.

Section 2 provides the needed background results on noncommutative stationary processes and their endomorphisms. Since spreadability immediately implies stationarity, most parts of the proof of Theorem 0.2 will be carried out in an equivalent framework of stationary processes.

We introduce in Section 3 two noncommutative versions of classical conditional independence, called 'conditional independence' (CI) and 'conditional factorizability' (CF). Both notions are equivalent if the conditioning is trivial or appropriate additional algebraic structure is supposed. But (CF) is a priori weaker than (CI) and more easily to control in applications. Their definition reflects that the conditioning is with respect to a von Neumann algebra which may *not* be contained in the image of two random variables. Further we relate 'conditional independence' to Popa's 'commuting squares' of von Neumann algebras.

The main result of Section 4 is that (CI) and (CF) are equivalent for a stationary random sequence if the conditioning is with respect to a subalgebra of the fixed point algebra of the corresponding endomorphism, see Theorem 4.2. Moreover, we introduce the notions of 'conditional order independence' ($\mathrm{CI_o}$) and 'conditional order factorizability' ($\mathrm{CF_o}$). These two notions are apparently weaker and reflect that the index set $\mathbb{N}_0$ of the random sequence is considered as an ordered set. Already ($\mathrm{CF_o}$), the weakest of the four properties, will suffice to establish mixing properties of stationary processes. Finally, we illustrate (CI) and (CF) by the algebraic reformulation of de Finetti's original example, an infinite sequence of exchangeable $\{0,1\}$-valued random variables.

Section 5 focuses on appropriate 'local perturbations' of $\mathbb{C}$-independent stationary random sequences. Our main result is that a noncommutative i.i.d. random sequence may be non-stationary. We provide related examples and disprove the implications '(d) $\Rightarrow$ (c)' and '($\mathrm{d_o}$) $\Rightarrow$ ($\mathrm{c_o}$)' of Theorem 0.2.

Section 6 provides a noncommutative generalization of Kolmogorov's zero-one-law for a random sequence with ($\mathrm{CF_o}$). Further we show in Theorem 6.4 that ($\mathrm{CF_o}$) and stationarity imply strong mixing over the tail algebra and fixed point characterization results. We coin in this section also the notion of a noncommutative Bernoulli shift, as it is suggested by our results on distributional symmetries and inspired by Kümmerer's notion of a generalized Bernoulli shift. These shifts can be recognized as the unilateral 'discrete' version of noncommutative continuous Bernoulli shifts from [HKK04].

Section 7 is devoted to an integral part of the noncommutative extended de Finetti theorem, the proof that spreadability of a random sequence yields conditional order independence ($\mathrm{CI_o}$). Here the conditioning is shown to be with respect to the tail algebra of the random sequence.

Section 8 upgrades the results of the previous section. Our main result is Theorem 8.1 which provides the proof of the crucial part of Theorem 0.2: spreadability implies conditional full independence (CI) of a random sequence. An important tool within its proof is a local version of the mean ergodic theorem, Theorem 8.4. It will allow us to perform mean ergodic approximations in a spreadability preserving manner.

Applications and an outview are contained in Section 9. We cite results from [GK08] on braidability and on the failure of the implications '(a) $\Leftarrow$ (b)' and '(b) $\Leftarrow$ (c)' of the noncommutative extended de Finetti theorem, Theorem 0.2. Moreover, we present a general central limit theorem for spreadable random sequences which can be regarded to be the noncommutative prototype of a conditioned central limit theorem. We also discuss briefly its potential connections to interacting Fock spaces. Finally, we give immediate applications of Theorem 0.2 to inequalities in noncommutative L^1-spaces, as they appear in the work of Junge and Xu.

Acknowledgments. The present paper took its origin from work with Rolf Gohm on one of the most simple examples coming from the Jones fundamental construction [GK08], and joint work with Roland Speicher on the structure of noncommutative white noises [KS07]. At both occasions we found 'spreadability' without being aware of it. We are indebted to Marius Junge and Wojciech Jaworski who independently pointed out possible connections to distributional symmetries and initiated the author's investigations resulting in the present paper. We are thankful to several helpful discussions with Benoit Collins, Rolf Gohm, Marius Junge, James Mingo and Roland Speicher in the course of writing this paper.

1. Preliminaries and Terminology

Noncommutative notions of probability spaces have in common that they consist of an algebra which is equipped with a linear functional. Here we shall work with the W*-algebraic version of such spaces, since they allow us to capture probabilistic tail events of random sequences. We refer the reader to [AFL82, Küm85, KM98, VDN92, NS06] for further information on noncommutative probability spaces, in particular *-algebraic or C*-algebraic settings.

Definition 1.1. A *probability space* $(\mathcal{M}, \psi)$ consists of a von Neumann subalgebra $\mathcal{M}$ with separable predual and a faithful normal state ψ on $\mathcal{M}$. A von Neumann algebra $\mathcal{M}_0$ of $\mathcal{M}$ is said to be *ψ-conditioned* if the ψ-preserving conditional expectation $E_{\mathcal{M}_0}$ from $\mathcal{M}$ onto $\mathcal{M}_0$ exists. Two probability spaces $(\mathcal{M}_1, \psi_1)$ and $(\mathcal{M}_2, \psi_2)$ are said to be *isomorphic* if there exists an isomorphism $\Pi\colon \mathcal{M}_1 \to \mathcal{M}_2$ such that $\psi_1 = \psi_2 \circ \Pi$. The ψ-preserving automorphisms of $\mathcal{M}$ will be denoted by $\mathrm{Aut}(\mathcal{M}, \psi)$.

By Takesaki's theorem, the ψ-preserving conditional expectation $E_{\mathcal{M}_0}$ exists if and only if $\sigma_t^\psi(\mathcal{M}_0) = \mathcal{M}_0$ for all $t \in \mathbb{R}$ [Tak03, IX, Theorem 4.2]. Here σ_t^ψ denotes the modular automorphism group associated to $(\mathcal{M}, \psi)$. Thus the existence of such a conditional expectation is automatic if ψ is a trace, i.e. $\psi(xy) = \psi(yx)$ for all $x, y \in \mathcal{M}$.

The noncommutative generalization of random variables is casted in terms of *-homomorphisms [AFL82]. For the purpose of this paper the following definition of a random variable will be sufficient.

Definition 1.2. Let $(\mathcal{A}_0, \varphi_0)$ and $(\mathcal{M}, \psi)$ be two probability spaces. A *random variable* is an injective *-homomorphism $\iota\colon \mathcal{A}_0 \to \mathcal{M}$ satisfying two additional properties:

(i) $\varphi_0 = \psi \circ \iota$;
(ii) $\iota(\mathcal{A}_0)$ is ψ-conditioned.

A random variable will also be addressed as the mapping $\iota\colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{M}, \psi)$.

Every classical random variable in the context of standard measure spaces yields by algebraisation a random variable in the sense of Definition 1.2. Conversely, if the von Neumann algebra $\mathcal{M}$ is commutative then the usual notion of a random variable on standard probability spaces can be recovered from Definition 1.2. Note that our assumption of injectivity is no restriction if a single random variable is considered.

Remark 1.3. Assertion (ii) in above definition is superfluous if ψ is a trace. Note also that this assertion has equivalent formulations:

(iii) ι intertwines the modular automorphism groups of $(\mathcal{A}_0, \varphi_0)$ and $(\mathcal{M}, \psi)$;
(iv) There exists a (unique) unital completely positive map $\iota^+\colon \mathcal{M} \to \mathcal{A}_0$ satisfying $\psi(x\iota(a)) = \varphi_0(\iota^+(x)a)$ for all $x \in \mathcal{M}$ and $a \in \mathcal{A}_0$.

The map ι^+ is also called the adjoint of ι. We refer the reader to [AC82, GK82, HKK04, AD06] for further details and background results on the equivalences of (ii) to (iv).

Remark 1.4. Commonly (selfadjoint) operators in the von Neumann algebra $\mathcal{M}$ (or, more generally, its noncommutative L^p-spaces) are also denoted as 'noncommutative random variables' in the literature. Such approaches require assumptions on the existence of higher moments of a random variable. The framework of injective *-homomorphisms has the advantage that it is free of any assumptions on the existence of moments. Of course, we can easily produce a random variable in the operator sense from our setting by considering $\iota(x)$ for some fixed $x \in \mathcal{A}_0$.

We are interested in sequences of random variables.

Notation 1.5. We write $I < J$ for two subsets $I, J \subset \mathbb{N}_0$ if $i < j$ for all $i \in I$ and $j \in J$. The cardinality of I is denoted by $|I|$. For $N \in \mathbb{N}$, we denote by $I + N$ the shifted set $\{i + N \mid i \in I\}$.

Definition 1.6. An *(identically distributed) random sequence* $\mathscr{I}$ is a sequence of random variables

$$\iota \equiv (\iota_i)_{i\in\mathbb{N}_0}\colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{M}, \psi).$$

The family $(\mathcal{A}_I)_{I\subset\mathbb{N}_0}$, with von Neumann subalgebras

$$\mathcal{A}_I = \bigvee_{i\in I} \iota_i(\mathcal{A}_0),$$

is called the *canonical filtration (generated by $\mathscr{I}$)* and $\mathscr{I}$ is said to be *minimal* if $\mathcal{A}_{\mathbb{N}_0} = \mathcal{M}$. The von Neumann subalgebra

$$\mathcal{A}^{\mathrm{tail}} := \bigcap_{n\in\mathbb{N}_0} \bigvee_{k\geq n} \iota_k(\mathcal{A}_0)$$

is called the *tail algebra of $\mathscr{I}$*.

Suppose a second random sequence $\widetilde{\mathscr{I}}$ is defined by the random variables $\widetilde{\iota} \equiv (\widetilde{\iota}_i)_{i\in\mathbb{N}_0} : (\mathcal{A}_0, \varphi_0) \to (\widetilde{\mathcal{M}}, \widetilde{\psi})$. Then $\mathscr{I}$ and $\widetilde{\mathscr{I}}$ are *isomorphic* if there exists an isomorphism $\Pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ such that $\psi \circ \Pi = \widetilde{\psi}$ and $\Pi \circ \widetilde{\iota}_n = \iota_n$ for all $n \in \mathbb{N}_0$.

Whenever it is convenient, we may turn a random sequence into a minimal one by restricting the probability space $(\mathcal{M}, \psi)$ to $(\mathcal{A}_{\mathbb{N}_0}, \psi|_{\mathcal{A}_{\mathbb{N}_0}})$. We have already introduced distributional symmetries in the introduction. Here we present equivalent definitions which are less intuitive, but more convenient within our proofs.

Notation 1.7. The group $\mathbb{S}_\infty$ is the inductive limit of the symmetric groups $\mathbb{S}_n$, $n \geq 2$, where $\mathbb{S}_n$ is generated on $\mathbb{N}_0$ by the transpositions

$$\pi_i : (i-1, i) \to (i, i-1)$$

with $1 \leq i < n$. By $[n]$ we denote the linearly ordered set $\{1, 2, \ldots, n\}$.

Definition 1.8. Let $\mathbf{i}, \mathbf{j} : [n] \to \mathbb{N}_0$ be two n-tuples.

(i) $\mathbf{i}$ and $\mathbf{j}$ are *translation equivalent*, in symbols: $\mathbf{i} \sim_\theta \mathbf{j}$, if there exists $k \in \mathbb{N}_0$ such that
$$\mathbf{i} = \theta^k \circ \mathbf{j} \qquad \text{or} \qquad \theta^k \circ \mathbf{i} = \mathbf{j}.$$
Here denotes θ the right translation $m \mapsto m+1$ on $\mathbb{N}_0$.

(ii) $\mathbf{i}$ and $\mathbf{j}$ are *order equivalent*, in symbols: $\mathbf{i} \sim_o \mathbf{j}$, if there exists a permutation $\pi \in \mathbb{S}_\infty$ such that
$$\mathbf{i} = \pi \circ \mathbf{j} \qquad \text{and} \qquad \pi|_{\mathbf{j}([n])} \text{ is order preserving.}$$

(iii) $\mathbf{i}$ and $\mathbf{j}$ are *symmetric equivalent*, in symbols: $\mathbf{i} \sim_\pi \mathbf{j}$, if there exists a permutation $\pi \in \mathbb{S}_\infty$ such that
$$\mathbf{i} = \pi \circ \mathbf{j}.$$

We have the implications $(\mathbf{i} \sim_\theta \mathbf{j}) \Rightarrow (\mathbf{i} \sim_o \mathbf{j}) \Rightarrow (\mathbf{i} \sim_\pi \mathbf{j})$.

Remark 1.9. Order equivalence of two n-tuples $\mathbf{i}$ and $\mathbf{j}$ can also be expressed with the help of the partial shifts $(\theta_N)_{N\geq 0} : \mathbb{N}_0 \to \mathbb{N}_0$, where

$$\theta_N(n) = \begin{cases} n & \text{if } n < N; \\ n+1 & \text{if } n \geq N. \end{cases}$$

Each θ_N is order-preserving and it is easy to see that $\mathbf{i} \sim_o \mathbf{j}$ if and only if there exist partial shifts $\theta_{N_1}, \theta_{N_2}, \ldots, \theta_{N_k}$ such that $\theta_{N_1} \circ \theta_{N_2} \circ \cdots \circ \theta_{N_k} \circ \mathbf{i} = \mathbf{j}$. Note also that any subsequence $(n_0, n_1, n_2, \cdots)$ of the infinite sequence $(0, 1, 2, 3, \ldots)$ can be approximated via actions of the subshifts $(\theta_N)_{N\geq 0}$.

Remark 1.10. Order equivalence is used in the context of a general limit theorem in [KS07] and our present formulation is an equivalent one.

For the notation of mixed higher moments of random variables, it is convenient to use Speicher's notation of multilinear maps.

Notation 1.11. Let the random sequence $\mathscr{I}$ be given by

$$\iota \equiv (\iota_i)_{i\in\mathbb{N}_0} : (\mathcal{A}_0, \varphi_0) \to (\mathcal{M}, \psi).$$

We put, for $\mathbf{i} : [n] \to \mathbb{N}_0$, $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{A}_0^n$ and $n \in \mathbb{N}$,

$$\begin{aligned} \iota[\mathbf{i}; \mathbf{a}] &:= \iota_{\mathbf{i}(1)}(a_1)\iota_{\mathbf{i}(2)}(a_2) \cdots \iota_{\mathbf{i}(n)}(a_n), \\ \psi_\iota[\mathbf{i}; \mathbf{a}] &:= \psi\big(\iota[\mathbf{i}; \mathbf{a}]\big). \end{aligned}$$

We are ready to introduce distributional symmetries in terms of the mixed moments of a random sequence.

Definition 1.12. *A random sequence $\mathscr{I}$ is*

(i) exchangeable *if, for any $n \in \mathbb{N}$,*

$$\psi_\iota[\mathbf{i}; \cdot\,] = \psi_\iota[\mathbf{j}; \cdot\,] \quad \textit{whenever} \quad \mathbf{i} \sim_\pi \mathbf{j};$$

(ii) spreadable *if, for any $n \in \mathbb{N}$,*

$$\psi_\iota[\mathbf{i}; \cdot\,] = \psi_\iota[\mathbf{j}; \cdot\,] \quad \textit{whenever} \quad \mathbf{i} \sim_o \mathbf{j};$$

(iii) stationary *if, for any $n \in \mathbb{N}$,*

$$\psi_\iota[\mathbf{i}; \cdot\,] = \psi_\iota[\mathbf{j}; \cdot\,] \quad \textit{whenever} \quad \mathbf{i} \sim_\theta \mathbf{j}.$$

We close this section with the proof of the obvious implications in the noncommutative extended de Finetti theorem.

Proof of Theorem 0.2, elementary parts. It is evident from Definition 1.8 and Definition 1.12 that exchangeability implies spreadability, and that spreadability implies stationarity. This shows the implication '(a) $\Rightarrow$ (b)' and the elementary parts on stationarity of the implications '(b) $\Rightarrow$ (c)' and '(b) $\Rightarrow$ (c_0)'. The implications '(c) $\Rightarrow$ (d)' and '(c_0) $\Rightarrow$ (d_0)' are trivial. □

2. Noncommutative stationary processes

Exchangeable or spreadable random sequences are stationary and can thus be expressed as stationary processes. Since the remaining sections of this paper will rest on this well known connection, we provide more in detail some of their specific properties in this section. We will introduce stationary processes such that they are in a canonical correspondence to stationary random sequences (see Definition 1.12). Their notion is very closely related to Kümmerer's approach in [Küm93, Küm03] (see also [Goh04, Section 2.1]). Moreover, we present a result from [Küm88a] which shows that a unilateral stationary process (as introduced next) extends to a bilateral stationary process.

Definition 2.1. A *(unilateral) stationary process $\mathscr{M}$* consists of a probability space $(\mathcal{M}, \psi)$, a ψ-conditioned subalgebra $\mathcal{M}_0 \subset \mathcal{M}$ and an endomorphism α of $\mathcal{M}$ satisfying

(i) unitality: $\alpha(\mathbb{1}) = \mathbb{1}$;
(ii) stationarity: $\psi \circ \alpha = \psi$;
(iii) conditioning: α and the modular automorphism group σ_t^ψ commute for all $t \in \mathbb{R}$.

The stationary process $\mathscr{M}$ is also denoted by the quadruple $(\mathcal{M}, \psi, \alpha, \mathcal{M}_0)$ and

$$\iota^\alpha \equiv (\iota_i^\alpha)_{i \in \mathbb{N}_0} \colon (\mathcal{M}_0, \psi_0) \to (\mathcal{M}, \psi), \qquad \iota_i^\alpha := \alpha^i|_{\mathcal{M}_0},$$

is called the *random sequence associated to $\mathscr{M}$*, for brevity also denoted by $\mathscr{I}^\alpha$.

The family of von Neumann subalgebras $(\mathcal{M}_I)_{I \subset \mathbb{N}_0}$, with

$$\mathcal{M}_I := \bigvee_{i \in I} \alpha^i(\mathcal{M}_0),$$

is called the *canonical filtration (generated by $\mathscr{M}$).* The von Neumann subalgebra

$$\mathcal{M}^{\mathrm{tail}} = \bigcap_{n\in\mathbb{N}_0} \alpha^n(\mathcal{M})$$

is called the *tail algebra of $\mathscr{M}$*. We denote by $\mathcal{M}^\alpha$ the *fixed point algebra* of the endomorphism α.

Finally, two minimal stationary processes $\mathscr{M}$ and $\widetilde{\mathscr{M}}$ are *isomorphic* if there exists an isomorphism $\Pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ such that

$$\psi \circ \Pi = \widetilde{\psi}, \quad \Pi \circ \widetilde{\alpha} = \alpha \circ \Pi, \quad \Pi(\widetilde{\mathcal{M}}_0) = \mathcal{M}_0.$$

The modular condition (iii) is needed for a non-tracial state ψ to ensure:

- ◇ Von Neumann algebras generated by the $\alpha^n(\mathcal{M}_0)$'s and their intersections are ψ-conditioned (see Remark 2.2).
- ◇ Stationary processes and stationary random sequences are in correspondence (see Lemma 2.5).
- ◇ A unilateral stationary process extends to a bilateral stationary process (see Theorem 2.7).

Remark 2.2. Condition (iii) of Definition 2.1 entails that the ψ-preserving conditional expectation $E_{\mathcal{M}_I}$ from $\mathcal{M}$ onto $\mathcal{M}_I$ exists: $\mathcal{M}_0$ is globally σ_t^ψ-invariant and now condition (iii) implies that $\alpha(\mathcal{M}_0)$ and, more generally, $\mathcal{M}_I$ are globally σ_t^ψ-invariant. Thus Takesaki's theorem on the existence of ψ-preserving conditional expectations applies. Of course, the condition (iii) can be dropped if ψ is a trace. We are indebted to Kümmerer for simple examples on hyperfinite III_λ factors such that $\alpha(\mathcal{A}_0)$ fails to be globally σ_t^ψ-invariant without condition (iii) [Küm].

To avoid reiterations throughout this paper we shall use the following convention for properties of a stationary process.

Definition 2.3. The stationary process $\mathscr{M}$ is said to have property 'A' if its associated random sequence $\mathscr{I}^\alpha$ has property 'A'. For example, $\mathscr{M}$ is minimal if its associated random sequence $\mathscr{I}^\alpha$ is minimal.

The canonical filtrations of a stationary process $\mathscr{M}$ and its associated random sequence $\mathscr{I}^\alpha$ always coincide. But the tail algebra $\mathscr{M}^{\mathrm{tail}}$ of $\mathscr{M}$ may be larger than the tail algebra of $\mathscr{I}^\alpha$,

$$\mathscr{M}^{\mathscr{I}\mathrm{tail}} = \bigcap_{n\in\mathbb{N}_0} \bigvee_{k\geq n} \iota_k^{(\alpha)}(\mathcal{M}_0) = \bigcap_{n\in\mathbb{N}_0} \bigvee_{k\geq n} \alpha^k(\mathcal{M}_0).$$

Lemma 2.4. *If $\mathscr{M}$ is minimal, then $\mathcal{M}^{\mathscr{I}\mathrm{tail}} = \mathcal{M}^{\mathrm{tail}}$.*

Proof. This is easily concluded from

$$\bigvee_{k\geq n} \iota_k^{(\alpha)}(\mathcal{M}_0) = \bigvee_{k\geq n} \alpha^k(\mathcal{M}_0) = \alpha^n \bigvee_{k\geq 0} \alpha^k(\mathcal{M}_0) \subseteq \alpha^n(\mathcal{M})$$

and minimality. □

We continue with the correspondence between stationary processes and stationary random sequences under the condition of minimality. We include this well known result for reasons of transparency since the proof of the noncommutative version of the extended de Finetti theorem makes heavily use of it.

Lemma 2.5. *There is a one-to-one correspondence between (equivalence classes of)*

(a) *minimal stationary processes* $\mathscr{M} = (\mathcal{M}, \psi, \alpha, \mathcal{M}_0)$*;*
(b) *minimal stationary random sequences* $\mathscr{I}$ *with random variables*

$$(\iota_n)_{n\geq 0} \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{M}, \psi).$$

The correspondence from (a) to (b) is given by

$$(\mathcal{A}_0, \varphi_0) := (\mathcal{M}_0, \psi|_{\mathcal{M}_0}) \quad \textit{and} \quad \iota_n := \alpha^n|_{\mathcal{M}_0}.$$

The correspondence from (b) to (a) is established via

$$\mathcal{M}_0 := \iota_0(\mathcal{A}_0) \quad \textit{and} \quad \alpha(\iota[\mathbf{i}; \mathbf{a}]) := \iota[\Theta \circ \mathbf{i}; \mathbf{a}]$$

for all $n \in \mathbb{N}$*, n-tuples* $\mathbf{i} \colon [n] \to \mathbb{N}_0$ *and* $\mathbf{a} \in \mathcal{A}_0^n$*.*

Proof. We omit all fairly clear parts of the proof and only show that the properties of $\mathscr{I}$ imply the modular condition $\alpha\sigma_t^{\psi} = \sigma_t^{\psi}\alpha$. Since the von Neumann algebras $\iota_n(\mathcal{A}_0)$ are ψ-conditioned, the random variables ι_n intertwine $\sigma_t^{\varphi_0}$ and σ_t^{ψ}, the modular automorphism groups of $(\mathcal{A}_0, \varphi_0)$ and $(\mathcal{M}, \psi)$ (see Remark 1.3 and [AD06, Lemma 2.5]). Thus

$$\begin{aligned} \sigma_t^{\psi} \circ \alpha(\iota[\mathbf{i}; \mathbf{a}]) &= \sigma_t^{\psi}\iota[\Theta \circ \mathbf{i}; \mathbf{a}] = \iota[\Theta \circ \mathbf{i}; \sigma_t^{\varphi_0}(\mathbf{a})] = \alpha(\iota[\mathbf{i}; \sigma_t^{\varphi_0}(\mathbf{a})]) \\ &= \alpha \circ \sigma_t^{\psi}(\iota[\mathbf{i}; \mathbf{a}]) \end{aligned}$$

establishes $\alpha\sigma_t^{\psi} = \sigma_t^{\psi}\alpha$ on a weak*-total subset of $\mathcal{M}$. Here $\sigma_t^{\varphi_0}(\mathbf{a})$ denotes the n-tuple $\big(\sigma_t^{\varphi_0}(a_1), \ldots \sigma_t^{\varphi_0}(a_n)\big)$. □

We will need the next theorem for approximations in the proof of Theorem 4.2.

Definition 2.6. A stationary process $\hat{\mathscr{M}} = (\hat{\mathcal{M}}, \hat{\psi}, \hat{\alpha}, \hat{\mathcal{M}}_0)$ is said to be *bilateral* if the endomorphism $\hat{\alpha}$ is an automorphism of $\hat{\mathcal{M}}$. A bilateral stationary process $\hat{\mathscr{M}}$ is *minimal* if $\hat{\mathcal{M}} = \bigvee_{n\in\mathbb{Z}} \hat{\alpha}^n(\hat{\mathcal{M}}_0)$.

Theorem 2.7. *A unilateral stationary process* $\mathscr{M} = (\mathcal{M}, \psi, \alpha, \mathcal{M}_0)$ *extends to a bilateral stationary process* $\hat{\mathscr{M}} = (\hat{\mathcal{M}}, \hat{\psi}, \hat{\alpha}, \hat{\mathcal{M}}_0)$*. In other words, there exists a random variable* $j \colon (\mathcal{M}, \psi) \to (\hat{\mathcal{M}}, \hat{\psi})$ *such that*

$$j(\mathcal{M}_0) = \hat{\mathcal{M}}_0 \quad \textit{and} \quad j\alpha^n = \hat{\alpha}^n j \qquad (n \in \mathbb{N}_0).$$

If $\hat{\mathscr{M}}$ *is minimal, then* $\hat{\mathcal{M}}^{\hat{\alpha}} = j(\mathcal{M}^{\alpha})$*.*

This theorem is immediate from Kümmerer's work on state-preserving Markov dilations. We provide some results from [Küm88a] which are essential for its proof.

Let $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ be two probability spaces. A morphism $T \colon (\mathcal{A}, \varphi) \to (\mathcal{B}, \psi)$ is a unital completely positive map $T \colon \mathcal{A} \to \mathcal{B}$ satisfying $\varphi = \psi \circ T$. The morphisms from $(\mathcal{A}, \varphi)$ into itself are denoted by $\mathrm{Mor}(\mathcal{A}, \varphi)$.

Definition 2.8. A morphism $T \in \mathrm{Mor}(\mathcal{A}, \varphi)$ admits a *state-preserving dilation* if there exists a probability space $(\hat{\mathcal{A}}, \hat{\varphi})$, an automorphism $\hat{T} \in \mathrm{Aut}(\hat{\mathcal{A}}, \hat{\varphi})$, two morphisms $j \colon (\mathcal{A}, \varphi) \to (\hat{\mathcal{A}}, \hat{\varphi})$ and $Q \colon (\hat{\mathcal{A}}, \hat{\varphi}) \to (\mathcal{A}, \varphi)$ such that $T^n = Q\,\hat{T}^n j$ for all $n \in \mathbb{N}_0$. A state-preserving dilation is *minimal* if $\hat{\mathcal{A}} = \bigvee_{n\in\mathbb{Z}} \hat{T}^n j(\mathcal{A})$.

Note in above definition that $T^n = Q\hat{T}^n j$ reads as $\mathrm{id}_{\mathcal{A}} = Qj$ for $n = 0$. This implies that j is a random variable from $(\mathcal{A}, \varphi)$ to $(\mathcal{B}, \psi)$ and the composition jQ is the ψ-preserving conditional expectation from $\mathcal{B}$ onto $j(\mathcal{A})$. We refer the reader to [Küm85] for further details.

Proposition 2.9 ([Küm88a]). *Let $(\mathcal{A}, \varphi)$ be a probability space and suppose α is a φ-preserving unital endomorphism of $\mathcal{A}$. Then the following are equivalent:*

(a) *α admits a state-preserving dilation.*
(b) *α commutes with the modular automorphism group σ_t^{φ}.*

We include the proof from [Küm88a] for the convenience of the reader. It uses inductive limits of C*-algebras (see for example [Sak71, Subsection 1.23]).

Proof. The implication '(a) ⇒ (b)' is shown in [Küm85, 2.1.8]. So it remains to prove the converse.

For $n \in \mathbb{N}_0$ put $\mathcal{A}_n := \mathcal{A}$ and for $n \geq m$ interpret α^{n-m} as a *-isomorphism of $\mathcal{A}_m$ into $\mathcal{A}_n$. Define $\tilde{\mathcal{A}}$ as the C*-inductive limit of $\{\mathcal{A}_m; \alpha^{n-m} \,|(n,m) \in \mathbb{N}_0 \times \mathbb{N}_0, n \geq m\}$. Moreover, putting $\varphi_n := \varphi$ for $n \in \mathbb{N}_0$, one has $\varphi_n(\alpha^{n-m}(x)) = \varphi_n(x)$ for $x \in \mathcal{A}_m = \mathcal{A} = \mathcal{A}_n$ with $n \geq m$. The state $\tilde{\varphi}$ on $\tilde{\mathcal{A}}$ is introduced as the inductive limit of $\{\varphi_m; \alpha^{n-m} \,|(n,m) \in \mathbb{N}_0 \times \mathbb{N}_0, n \geq m\}$ [Sak71, 1.23.10]. Identify $\tilde{\mathcal{A}}$ with the norm closure of $\bigcup_{n\geq 0} \mathcal{A}_n$. Thus $\alpha(\mathcal{A}_n)$ is identified with $\mathcal{A}_{n-1}$ $(n \geq 1)$. Consequently α extends to $\bigcup_{n\geq 0} \mathcal{A}_n$ and then to its norm closure $\tilde{\mathcal{A}}$. Doing so we obtain a $\tilde{\varphi}$-preserving automorphism $\tilde{\alpha}$ of $\tilde{\mathcal{A}}$. We define an injection $\tilde{j}$ by identifying $\mathcal{A}$ with $\mathcal{A}_0$.

Since α commutes with the modular automorphism group σ_t^{φ} the subalgebra $\alpha^n(\mathcal{A}) \subset \mathcal{A}$ is globally σ_t^{φ}-invariant and the φ-preserving conditional expectation from $\mathcal{A}$ onto $\mathcal{A}_n$ exists for all $n \in \mathbb{N}$ (see [Tak03, Theorem]). Correspondingly, for each $n \in \mathbb{N}$, we find a completely positive map $Q_n \colon \mathcal{A}_n \to \mathcal{A}$ such that, for $m \leq n$,

$$\varphi_n = \varphi \circ Q_n, \qquad Q_n \circ \tilde{j} = \mathrm{id}_{\mathcal{A}}, \qquad Q_n|_{\mathcal{A}_m} = Q_m.$$

By continuity this leads to a completely positive map $\tilde{Q} \colon \tilde{\mathcal{A}} \to \mathcal{A}$ such that, for $n \geq 1$,

$$\tilde{\varphi} = \varphi \circ \tilde{Q}, \qquad \tilde{Q} \circ \tilde{j} = \mathrm{id}_{\mathcal{A}}, \qquad \tilde{Q}|_{\mathcal{A}_n} = Q_n.$$

Let $\sigma_t^{\varphi_m}$ be the modular automorphism group associated to $(\mathcal{A}_m, \varphi_m)$. It follows $\sigma_t^{\varphi_n}(x) = \sigma_t^{\varphi_m}(x)$ for $x \in \mathcal{A}_m$, $n \geq m$. Therefore the modular groups on $\mathcal{A}_n$ extend to a group σ_t on $\tilde{\mathcal{A}}$ such that $\tilde{\varphi}$ satisfies the KMS condition with respect to σ_t (see [Ped79, 8.12.3]). Hence $\tilde{\varphi}$ extends to a faithful normal state $\hat{\varphi}$ on $\hat{\mathcal{A}} := \Pi_{\tilde{\varphi}}(\tilde{\mathcal{A}})''$ [Ped79, 8.14.4].

Now it is routine to show that $\tilde{\alpha}$ extends to the $\hat{\varphi}$-preserving automorphism $\hat{\alpha}$ of $\hat{\mathcal{A}}$, the map $\tilde{Q}$ to the completely positive map $Q \colon \hat{\mathcal{A}} \to \mathcal{A}$ satisfying $\varphi \circ Q = \hat{\varphi}$ and the injection $\tilde{j}$ to an injective *-homomorphism $j \colon \mathcal{A} \to \hat{\mathcal{A}}$ such that $\varphi = \hat{\varphi} \circ j$ and $j(\mathcal{A}) = \Pi_{\tilde{\varphi}}(\mathcal{A}_0)''$. Finally, $\alpha^n = Q\hat{\alpha}^n j$ is immediately verified for $n \in \mathbb{N}_0$. □

Proof of Theorem 2.7. The endomorphism α of $\mathscr{M}$ satisfies the condition (b) of Proposition 2.9. Thus there exists a probability space $(\hat{\mathcal{M}}, \hat{\psi})$, a $\hat{\psi}$-preserving automorphism $\hat{\alpha}$ of $\hat{\mathcal{M}}$, and a random variable $j \colon (\mathcal{M}, \psi) \to (\hat{\mathcal{M}}, \hat{\psi})$ such that $j\alpha^n = \hat{\alpha}^n j$ for all $n \in \mathbb{N}_0$. Clearly $\hat{\mathcal{M}}_0 := j(\mathcal{M}_0)$ is a $\hat{\psi}$-conditioned subalgebra of $\hat{\mathcal{M}}$. Finally, $\hat{\mathcal{M}}^{\hat{\alpha}} = j(\mathcal{M}^{\alpha})$ is the content of [Küm85, Corollary 3.1.4]. □

3. Conditional independence and conditional factorizability

From our investigations of distributional symmetries emerge two closely related noncommutative generalizations of classical conditional independence. Here we concentrate on the case of two random variables; the more general setting of random sequences is covered in the consecutive section where we will meet a further ramification of these two notions.

Definition 3.1. Let $(\mathcal{M}, \psi)$ be a probability space with three ψ-conditioned von Neumann subalgebras $\mathcal{M}_0, \mathcal{M}_1$ and $\mathcal{M}_2$. Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are said to be

(i) $\mathcal{M}_0$*-independent* or *conditionally independent* if
$$E_{\mathcal{M}_0}(xy) = E_{\mathcal{M}_0}(x)E_{\mathcal{M}_0}(y)$$
for all $x \in \mathcal{M}_1 \vee \mathcal{M}_0$ and $y \in \mathcal{M}_2 \vee \mathcal{M}_0$;

(ii) $\mathcal{M}_0$*-factorizable* or *conditionally factorizable* if
$$E_{\mathcal{M}_0}(xy) = E_{\mathcal{M}_0}(x)E_{\mathcal{M}_0}(y)$$
for all $x \in \mathcal{M}_1$ and $y \in \mathcal{M}_2$.

This definition does *not* assume the inclusion $\mathcal{M}_0 \subset \mathcal{M}_1 \cap \mathcal{M}_2$. It is open if conditional factorizability implies conditional independence and thus the equivalence of these two notions. But this is of course the case if $\mathcal{M}_0 \simeq \mathbb{C}$, and we will state in Lemma 3.6 further conditions under which $\mathcal{M}_0$-factorizability implies $\mathcal{M}_0$-independence.

Remark 3.2. An alternative formulation of $\mathcal{M}_0$-independence is sometimes easier to verify in applications. Under the assertions of Definition 3.1, the following are equivalent:

(a) $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{M}_0$-independent;

(b) there exist $\mathcal{M}_0$-independent von Neumann subalgebras $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ of $\mathcal{M}$ such that $\mathcal{M}_0 \subset \widetilde{\mathcal{M}}_1 \cap \widetilde{\mathcal{M}}_2$ and $\mathcal{M}_i \subset \widetilde{\mathcal{M}}_i$ $(i = 1, 2)$.

Since this equivalence is fairly clear, we omit its proof.

Remark 3.3. If $\mathcal{M}_0 \simeq \mathbb{C}$, we will also write $\mathbb{C}$-independence instead of $\mathcal{M}_0$-independence. Note that $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathbb{C}$-independent if and only if $\psi(xy) = \psi(x)\psi(y)$ for all $x \in \mathcal{M}_1$ and $y \in \mathcal{M}_2$ [Küm88b].

The failure of the inclusion $\mathcal{M}_0 \subset \mathcal{M}_1 \cap \mathcal{M}_2$ happens frequently in the context of distributional symmetries and is, in classical probability, intimately related to random probability measures. We illustrate this by the most simple example which may be taken from classical probability (just choose $\mathcal{A} \simeq \mathbb{C}^2 \otimes \mathbb{C}^2$ in Example 3.4).

Example 3.4. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two $\mathbb{C}$-independent von Neumann subalgebras of the probability space $(\mathcal{A}, \varphi)$. We define the probability space $(\mathcal{M}, \psi)$ by $\mathcal{M} := \mathcal{A} \oplus \mathcal{A}$ and $\psi := \frac{1}{2}(\varphi \oplus \varphi)$. For $i = 1, 2$, the embeddings $\mathcal{A}_i \ni x \to x \oplus x \in \mathcal{M}$ define the von Neumann subalgebras $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Furthermore, we put $\mathcal{M}_0 = \mathbb{C}\mathbb{1}_{\mathcal{A}} \oplus \mathbb{C}\mathbb{1}_{\mathcal{A}} \simeq \mathbb{C}^2$. One has $\mathcal{M}_i \vee \mathcal{M}_0 = \mathcal{A}_i \oplus \mathcal{A}_i$ for $i = 1, 2$ and calculates
$$E_{\mathcal{M}_0}(xy) = E_{\mathcal{M}_0}(x)E_{\mathcal{M}_0}(y)$$
for all $x \in \mathcal{M}_1 \vee \mathcal{M}_0$ and $y \in \mathcal{M}_2 \vee \mathcal{M}_0$. Thus $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{M}_0$-independent. But $\mathcal{M}_1 \cap \mathcal{M}_2 \simeq \mathbb{C}$, so $\mathcal{M}_0 \not\subset \mathcal{M}_1 \cap \mathcal{M}_2$.

Remark 3.5. Another calculation shows in the above example that $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathbb{C}$-independent. But this is rather an accident because we have chosen identical states on each component of the direct sum.

Evidently, $\mathcal{M}_0$-independence implies $\mathcal{M}_0$-factorizability. But it is open if $\mathcal{M}_0$-factorizability implies $\mathcal{M}_0$-independence. Frequently this can be concluded if additional algebraic structures are available (see also Theorem 4.2). All presently known examples (within our setting) satisfy at least one of the following conditions.

Lemma 3.6. *$\mathcal{M}_0$-factorizability and $\mathcal{M}_0$-independence are equivalent under each of the following additional assertions:*

(i) *(trivial conditioning) $\mathcal{M}_0 \simeq \mathbb{C}$;*
(ii) *(central conditioning) $\mathcal{M}_0 \subset \mathcal{M} \cap \mathcal{M}'$;*
(iii) *(classical probability) $\mathcal{M} = \mathcal{M}'$;*
(iv) *(relative commutants) $\mathcal{M}_0 \subset {\mathcal{M}_1}' \cap {\mathcal{M}_2}'$;*
(v) *(commuting squares) $\mathcal{M}_0 \subset \mathcal{M}_1 \cap \mathcal{M}_2$.*

Proof. Each of the assertions (i) to (iv) implies that the vector spaces

$$\{ax \,|\, a \in \mathcal{M}_0, x \in \mathcal{M}_1\} \quad \text{and} \quad \{yb \,|\, b \in \mathcal{M}_0, y \in \mathcal{M}_2\}$$

are weak* total in $\mathcal{M}_0 \vee \mathcal{M}_1$ and $\mathcal{M}_0 \vee \mathcal{M}_2$, respectively. Thus the module property of conditional expectations and $\mathcal{M}_0$-factorizability imply

$$E_{\mathcal{M}_0}(axyb) = aE_{\mathcal{M}_0}(xy)b = aE_{\mathcal{M}_0}(x)E_{\mathcal{M}_0}(y)b = E_{\mathcal{M}_0}(ax)E_{\mathcal{M}_0}(yb).$$

This equalities extend bilinearly and an approximation argument completes the proof in the cases (i) to (iv). The proof under the assertion (v) is trivial. □

Our notion of conditional independence is in close contact with Popa's notion of *commuting squares* [Pop83a, Pop83b, PP86]. Detailed information on their role in subfactor theory is provided in [JS97, GHJ89]. We will make frequent use of some of their properties. Note that these assertions do *not* apply for conditional factorizability.

Proposition 3.7. *Suppose $\mathcal{M}_0 \subset \mathcal{M}_1 \cap \mathcal{M}_2$, in addition to the assertions of Definition 3.1. Then the following are equivalent:*

(i) *$\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{M}_0$-independent;*
(ii) *$E_{\mathcal{M}_1}(\mathcal{M}_2) = \mathcal{M}_0$;*
(iii) *$E_{\mathcal{M}_1}E_{\mathcal{M}_2} = E_{\mathcal{M}_0}$*
(iv) *$E_{\mathcal{M}_1}E_{\mathcal{M}_2} = E_{\mathcal{M}_2}E_{\mathcal{M}_1}$ and $\mathcal{M}_1 \cap \mathcal{M}_2 = \mathcal{M}_0$.*

In particular, it holds $\mathcal{M}_0 = \mathcal{M}_1 \cap \mathcal{M}_2$ if one and thus all of the four assertions are satisfied.

Proof. The tracial case for ψ is proved in [GHJ89, Prop. 4.2.1]. The non-tracial case follows from this, after some minor modifications of the arguments therein. □

We close this section with some remarks on examples and references which are closely related to conditional independence in our noncommutative setting. The author is presently not aware of published examples in the quantum setting beyond the assertions stated in Lemma 3.6. It would be of interest to find examples of von Neumann algebras which are conditionally factorizable, but not conditionally independent, if possible at all.

Remarks 3.8. (1) $\mathbb{C}$-independence emerged from investigations of Kümmerer on the structure of stationary quantum Markov processes [Küm85, Küm88a, Küm88b, Küm93]. Its generalization to commuting squares is explored further from the perspective of noncommutative probability in [Rup95, Kös00, Kös03, HKK04, KS07].
(2) Examples for $\mathbb{C}$-independence are classical independence, tensor independence and free independence. Further examples originate from pioneering work of Bożejko and Speicher [BS91, BS94] and are given by generalized or noncommutative Gaussian random variables [BKS97, GM02, Kró02]. The most well known among them are q-Gaussian random variables. Crucial for the appearance of $\mathbb{C}$-independence are the presence of white noise functors [Küm96, GM02] and a vacuum vector of the underlying deformed Fock space which is separating for the considered von Neumann algebras.
(3) Sources of examples for $\mathcal{M}_0$-independence are, of course, conditional independence in probability theory and random probability measures on standard Borel probability spaces. Further examples, satisfying the inclusion $\mathcal{M}_0 \subset \mathcal{M}_1 \cap \mathcal{M}_2$, arise from amplifications of examples for $\mathbb{C}$-independence by tensor product constructions. Freeness with amalgamation as well as commuting squares from subfactor theory are further sources of $\mathcal{M}_0$-independence (with $\mathcal{M}_0 \subset \mathcal{M}_1 \cap \mathcal{M}_2$). We refer to [HKK04] for a more detailed treatment of some of these examples.
(4) $\mathcal{M}_0$-independence appears, also under the assumption $\mathcal{M}_0 \subset \mathcal{M}_1 \cap \mathcal{M}_2$, in the work of Junge and Xu on noncommutative Rosenthal inequalities [JX03] and within Junge's quantum probabilistic approach to embedding Pisier's operator Hilbert space OH into the predual of the hyperfinite III_1-factor [Jun06].

4. Stationarity and conditional independence/factorizability

This section is devoted to show in Theorem 4.2 that conditional factorizability implies conditional independence in the context of stationarity and under a certain conditioning. We close with an illustration of conditional independence and conditional factorizability by an algebraic treatment of an infinite sequence of exchangeable $\{0,1\}$-valued random variables.

Due to the noncommutativity of our setting, there are (at least) two natural ways to extend the notions of conditional independence and conditional factorizablility (see Definition 3.1) from two random variables to random sequences indexed by $\mathbb{N}_0$. One may regard $\mathbb{N}_0$ as a set, or as an ordered set (with its natural order).

Definition 4.1. The (identically distributed) random sequence $\mathscr{I}$, given by

$$\iota \equiv (\iota_i)_{i\in\mathbb{N}_0} \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{M}, \psi),$$

with canonical filtration $(\mathcal{A}_I)_{I\subset\mathbb{N}_0}$, is said to be

- (CI) *full $\mathcal{N}$-independent* or *conditionally full independent*, if $\mathcal{A}_I$ and $\mathcal{A}_J$ are $\mathcal{N}$-independent for all $I, J \subset \mathbb{N}_0$ with $I \cap J = \emptyset$;
- ($\mathrm{CI_o}$) *order $\mathcal{N}$-independent* or *conditionally order independent*, if $\mathcal{A}_I$ and $\mathcal{A}_J$ are $\mathcal{N}$-independent for all $I, J \subset \mathbb{N}_0$ with $I < J$ or $I > J$;

We say that $\mathscr{I}$ is

- (CF) *full $\mathcal{N}$-factorizable* or *conditionally full factorizable*, if $\mathcal{A}_I$ and $\mathcal{A}_J$ are $\mathcal{N}$-factorizable for all $I, J \in\subset \mathbb{N}_0$ with $I \cap J = \emptyset$;
- ($\mathrm{CF_o}$) *order $\mathcal{N}$-factorizable* or *conditionally order factorizable*, if $\mathcal{A}_I$ and $\mathcal{A}_J$ are $\mathcal{N}$-factorizable for all $I, J \subset \mathbb{N}_0$ with $I < J$ or $I > J$;

We will deliberately drop the attributes 'full' or 'order' if we want to address conditional independence or conditional factorizability only on the informal level or if it is clear from the context whether the index set $\mathbb{N}_0$ is regarded with order structure or without it. Obviously we have the following implications:

$$\begin{array}{ccc} (\mathrm{CI}) & \Rightarrow & (\mathrm{CF}) \\ \Downarrow & & \Downarrow \\ (\mathrm{CI_o}) & \Rightarrow & (\mathrm{CF_o}). \end{array}$$

We record that this gives the following implications in the noncommutative extended de Finetti theorem.

Proof of Theorem 0.2, (c) ⇒ (c_o) and (d) ⇒ (d_o). This is obvious for $\mathcal{N} = \mathcal{M}^{\mathrm{tail}}$ from Definition 4.1 and above diagram. □

A natural question is to ask if the converse implications in above diagram are also valid. Actually, we do not know an answer in this generality. But an affirmative answer is available for the equivalence of conditional independence and conditional factorizability if the random sequence $\mathscr{I}$ is stationary and $\mathcal{N}$ contained in the fixed point algebra of the corresponding stationary process (see Lemma 2.5 for this correspondence).

Theorem 4.2. *Let $\mathscr{M}$ be a minimal stationary process and suppose the ψ-conditioned von Neumann subalgebra $\mathcal{N}$ satisfies $\mathcal{N} \subset \mathcal{M}^{\alpha}$. Then the following are equivalent:*

(CI) *$\mathscr{M}$ is full $\mathcal{N}$-independent;*
(CF) *$\mathscr{M}$ is full $\mathcal{N}$-factorizable.*

Furthermore, the following are equivalent under the same assertions:

($\mathrm{CI_o}$) *$\mathscr{M}$ is order $\mathcal{N}$-independent;*
($\mathrm{CF_o}$) *$\mathscr{M}$ is order $\mathcal{N}$-factorizable.*

We will see in Section 6 that conditional order factorizability ($\mathrm{CF_o}$), the weakest of the four properties, already suffices to identify $\mathcal{N}$ as the fixed point algebra of the endomorphism α which equals moreover the tail algebra. There it will suffice, due to Theorem 4.2, to establish these fixed point characterization results of Kolmogorov type on the level of conditional factorizability. Moreover we will benefit from this simplification in Section 7 and Section 8 when showing that spreadability implies conditional independence.

We prepare the proof of Theorem 4.2 by some well known results on approximations.

Lemma 4.3. *Let $x_1, \ldots, x_p \in \mathrm{B}_1(\mathcal{M})$, the unit ball of $\mathcal{M}$. Suppose further that each x_i is approximated by a sequence $(x_{i,n})_{n\in\mathbb{N}} \subset \mathrm{B}_1(\mathcal{M})$ in the strong operator topology. Then*

$$x_1 x_2 \cdots x_p = \text{SOT-}\lim_{n\to\infty} x_{1,n} x_{2,n} \cdots x_{p,n}.$$

Proof. This is evident from the definition of the strong operator topology in the case $p = 2$, since $\|x_{1,n}\| \leq 1$ and

$$x_{1,n}x_{2,n} - x_1x_2 = x_{1,n}(x_{2,n} - x_2) + (x_{1,n} - x_1)x_2.$$

The more general case $p > 2$ is concluded by induction. □

Lemma 4.4. *Suppose $\mathscr{M}$ is a minimal bilateral stationary process and let the function $N\colon \mathbb{N} \to \mathbb{Z}$ be given. Then every $a \in \mathcal{M}^\alpha$ is approximated by a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ in the strong operator topology such that*

$$a_n \in \mathcal{M}_{\{0,1,\ldots,n-1\}+N(n)} \quad \textit{and} \quad \|a_n\| \leq \|a\|.$$

Proof. We assume without loss of generality that all a is in the unit ball $\mathrm{B}_1(\mathcal{M}) \cap \mathcal{M}^\alpha$. The *-algebra $\mathcal{M}_{\mathbb{N}_0}^{\mathrm{alg}}$ is weak*-dense in $\mathcal{M}$. Thus, by Kaplansky's density theorem, $a \in \mathcal{N}$ is approximated by a sequence $(b_n)_n \subset \mathcal{M}_{\mathbb{N}_0}^{\mathrm{alg}} \cap \mathrm{B}_1(\mathcal{M})$ in the strong operator topology. Put

$$a_n := \alpha^{N(n)} E_{\mathcal{M}_{[0,n-1]}}(b_n) \in \mathcal{M}_{\{0,1,\ldots,n-1\}+N(n)}.$$

and note that α is an automorphism of $\mathcal{M}$, since we are working in the bilateral setting. We claim that

$$\text{SOT-}\lim_n a_n = a. \tag{4.1}$$

Indeed, the sequence $(E_{\mathcal{M}_{[0,n-1]}})_n$ is norm bounded and converges to $\mathrm{id}_{\mathcal{M}}$ in the pointwise strong operator topology; this is clear on $\mathcal{M}_{\mathbb{N}_0}^{\mathrm{alg}}$ and an $\frac{\varepsilon}{2}$-argument gives the general case. Thus $\big(E_{\mathcal{M}_{[0,n-1]}}(b_n)\big)_n$ converges to a in the strong operator topology. We use next the ψ-topology which is induced by the maps $\mathcal{M} \ni x \mapsto \psi(x^*x)^{1/2}$. Since the strong operator topology and the ψ-topology coincide on bounded sets,

$$\begin{aligned} \|a_n - a\|_\psi &= \|\alpha^{N(n)}\big(E_{\mathcal{M}_{[0,n-1]}}(b_n) - a\big)\|_\psi \\ &= \|E_{\mathcal{M}_{[0,n-1]}}(b_n) - a\|_\psi \end{aligned}$$

completes the proof. □

Proof of Theorem 4.2. Only the implications '($\mathrm{CF_o}$) $\Longrightarrow$ ($\mathrm{CI_o}$)' and '(CF) $\Longrightarrow$ (CI)' require a proof, since their reverse implications are trivial. We can assume by Theorem 2.7 that $\mathscr{M} = (\mathcal{M}, \psi, \alpha, \mathcal{M}_0)$ is a minimal bilateral stationary process. This will allow us to approximate elements of $\mathscr{N}$ in an appropriate manner. Note that full (resp. order) $\mathcal{N}$-factorizability of the family $(\mathcal{M}_I)_{I\subset\mathbb{N}_0}$ implies immediately full (resp. order) $\mathcal{N}$-factorizability of $(\mathcal{M}_I)_{I\subset\mathbb{Z}}$ by stationarity; this is clear for finite sets I and the general case is done by approximation.

We need to show that full (resp. order) $\mathcal{N}$-factorizability of $(\mathcal{M}_I)_{I\subset\mathbb{N}}$ implies

$$E_{\mathcal{N}}(xy) = E_{\mathcal{N}}(x)E_{\mathcal{N}}(y)$$

for all $x \in \mathcal{M}_I \vee \mathcal{N}$ and $y \in \mathcal{M}_J \vee \mathcal{N}$ with $I \cap J = \emptyset$ (resp. $I < J$).

For this purpose, we start with bounded sets $I, J \subset \mathbb{N}_0$ and consider monomials of the form

$$x = z_1 a_1 \cdots z_p a_p \quad \text{and} \quad y = z_{p+1} a_{p+1} \cdots z_{2p} a_{2p},$$

with $z_i \in \mathcal{M}_I$, $z_{p+i} \in \mathcal{M}_J$ and $a_i, a_{i+p} \in \mathcal{N}$ $(i = 1, \ldots, p)$. We approximate all a_i's in the strong operator topology and can assume without loss of generality that all z_i's and a_i's are in the unit ball $\mathrm{B}_1(\mathcal{M})$. Let $N_i\colon \mathbb{N} \to \mathbb{Z}$ be given function which will be specified later. By Lemma 4.4, there exist sequences $(a_{i,n})_{n\in\mathbb{N}} \subset \mathrm{B}_1(\mathcal{M})$ satisfying

$$\begin{aligned} a_i &= \text{SOT-}\lim_{n\to\infty} a_{i,n}, \\ a_{i,n} &\in \mathcal{M}_{\{0,1,\ldots,n-1\}+N_(n)}. \end{aligned}$$

Let

$$\begin{aligned} x_n &:= z_1 a_{1,n} z_2 a_{2,n} \cdots z_p a_{p,n}, \\ y_n &:= z_{p+1} a_{p+1,n} z_{p+2} a_{p+2,n} \cdots z_{2p} a_{2p,n}. \end{aligned}$$

We specify next the choice of the functions N_i. Let $N_i(n) := -n$ and $N_{p+i}(n) := N$ for $i = 1, \dots, p$, where $N > \max I \cup J$. Note that the sets

$$\begin{aligned} I_n &:= I \cup \{-n, n+1, \dots, -1\}, \\ J_n &:= J \cup \{N, N+1, \dots, N+n-1\} \end{aligned}$$

are disjoint if I and J are disjoint; and that $I_n < J_n$ if $I < J$. Since $x_n \in \mathcal{M}_{I_n}$ and $y_n \in \mathcal{M}_{J_n}$ we conclude from order (resp. full) $\mathcal{N}$-factorizability that

$$E_{\mathcal{N}}(x_n y_n) = E_{\mathcal{N}}(x_n) E_{\mathcal{N}}(y_n),$$

which entails

$$\begin{aligned} &E_{\mathcal{N}}(xy) - E_{\mathcal{N}}(x) E_{\mathcal{N}}(y) \\ &= E_{\mathcal{N}}(xy - x_n y_n) + E_{\mathcal{N}}(x_n) E_{\mathcal{N}}(y_n - y) + E_{\mathcal{N}}(x_n - x) E_{\mathcal{N}}(y). \end{aligned}$$

We infer from Lemma 4.3 and the SOT-SOT-continuity of conditional expectations that the right hand side of this equation vanishes for $n \to \infty$ in the strong operator topology. Thus full (resp. order) $\mathcal{N}$-factorizability implies, for each $p \in \mathbb{N}$,

$$\begin{aligned} &E_{\mathcal{N}}(z_1 a_1 \cdots z_p a_p z_{p+1} a_{p+1} \cdots z_{2p} a_{2p}) \\ &\quad = E_{\mathcal{N}}(z_1 a_1 \cdots z_p a_p) E_{\mathcal{N}}(z_{p+1} a_{p+1} \cdots z_{2p} a_{2p}) \end{aligned} \tag{4.2}$$

for any $z_1, \dots, z_p \in \mathcal{M}_I$, $z_{p+1}, \cdots z_{2p} \in \mathcal{M}_J$ and $a_1, \cdots a_{2p} \in \mathcal{N}$ whenever I and J are disjoint (resp. ordered) and bounded. This equality extends by bilinearity to the *-algebras $\mathcal{M}_I \cup \mathcal{N}$ and $\mathcal{M}_J \cup \mathcal{N}$. (By filling in additional factors $\mathbb{1}_{\mathcal{M}}$ if necessary we can always achieve that monomials have the same number of factors.) Since $\mathcal{M}_I \cup \mathcal{N}$ and $\mathcal{M}_J \cup \mathcal{N}$ are weak* dense in $\mathcal{M}_I \vee \mathcal{N}$ and $\mathcal{M}_J \vee \mathcal{N}$, the equality (4.2) extends further to the weak* closure, using Kaplansky's density theorem and arguments similar to that in the proof of Theorem 6.4. Finally, another density argument extends the validity of (4.2) from bounded disjoint sets I and J to possibly unbounded disjoint sets. □

Remark 4.5. At the time of this writing and in the generality of our setting, we have no information about the validity of the remaining implications $(\mathrm{CI_o}) \Rightarrow (\mathrm{CI})$ and $(\mathrm{CF_o}) \Rightarrow (\mathrm{CF})$, even under the assumptions of stationarity and $\mathcal{N} \simeq \mathbb{C}$. In particular, we do not know if an infinite stationary random sequence exists which is conditionally order independent, but fails to be conditionally full independent.

We continue with an illustration of above concepts of conditional independence and conditional factorizability for stationary random sequences. The example is the von Neumann algebraic reformulation of infinite sequence of zero-one-valued random variables, as they have been the subject of de Finetti's pioneering investigations on exchangeability [Fin31]. We will observe in this example why it is too restrictive to assume that $\mathcal{N}$ is contained in the image of random variables.

Example 4.6. Let $(\mathcal{A}_0, \varphi_0)$ be given by

$$\mathcal{A}_0 = \mathbb{C}^2 \quad \text{and} \quad \varphi_0 = \mathrm{tr}_p$$

with $\operatorname{tr}_p((a_1, a_2)) = pa_1 + (1-p)a_2$ for some fixed $p \in (0,1)$. We realize the probability space $(\mathcal{M}, \varphi)$ as a mixture of infinite coin tosses with respect to some probability measure ν on the standard measurable space $([0,1], \Sigma)$, assuming $\nu(\{0\}) = \nu(\{1\}) = 0$ and $\nu(\{p\}) < 1$ for any $p \in (0,1)$:

$$\mathcal{M} = \int_{[0,1]}^{\oplus} \mathcal{M}(p) d\nu(p), \qquad \mathcal{M}(p) = \bigotimes_{n \in \mathbb{N}_0} \mathbb{C}_2,$$
$$\psi = \int_{[0,1]}^{\oplus} \psi(p) d\nu(p), \qquad \psi(p) = \bigotimes_{n \in \mathbb{N}_0} \operatorname{tr}_p .$$

Here denotes $\mathcal{M}(p)$ the infinite von Neumann algebraic tensor product of $\mathbb{C}_2$ with respect to the infinite tensor product state on $\psi(p)$ which are obtained by passing through the GNS construction starting from the *-algebra $\bigcup_{k\in\mathbb{N}} \otimes_{k=0}^n \mathbb{C}_2$ equipped with the product state $\bigcup_{k\in\mathbb{N}} \otimes_{k=0}^n \operatorname{tr}_p$. We refer the reader to [Tak79] for further information on direct integrals of von Neumann algebras and states.
The random variable $\iota_i \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{M}, \psi)$, with $i \in \mathbb{N}_0$, is defined by the constant embedding of $a \in \mathbb{C}_2$ into the i-th factor of each fiber of the direct integral:

$$\iota_i(a) = \int_{[0,1]}^{\oplus} \underbrace{\mathbb{1}_{\mathcal{A}_0} \otimes \cdots \otimes \mathbb{1}_{\mathcal{A}_0}}_{i \text{ factors}} \otimes\, a \otimes \mathbb{1}_{\mathcal{A}_0} \otimes \cdots d\nu(p)$$

Finally, we put

$$\mathcal{N} := \int_{[0,1]}^{\oplus} \mathbb{C}\mathbb{1}_{\mathcal{M}(p)} d\nu(p) \simeq L^\infty([0,1], \nu).$$

Note that our assumptions on the measure ν imply $\mathcal{N} \not\simeq \mathbb{C}$.
The canonical filtration $(\mathcal{A}_I)_{I \subset \mathbb{N}_0}$ generated by the random sequence $\iota \equiv (\iota_i)_{i \in \mathbb{N}_0}$ is defined by

$$\mathcal{A}_I = \bigvee_{i \in I} \iota_i(\mathcal{A}_0).$$

The random sequence ι is minimal, i.e. we have

$$\mathcal{M} = \bigvee_{n \in \mathbb{N}_0} \iota_i(\mathcal{A}_0).$$

This follows if we can ensure that $\bigvee_{n\in\mathbb{N}_0} \iota_i(\mathcal{A}_0)$ contains $\mathcal{N}$. Indeed, Kakutani's theorem entails that the family of infinite product states $\{\psi(p)\}_{p\in(0,1)}$ is mutually disjoint [Hid80]. We conclude from this that every element $x \in \mathcal{N} \simeq L^\infty([0,1],\nu)$ can be approximated by a bounded sequence $(x_n)_{n\in\mathbb{N}} \subset \bigcup_{i\in\mathbb{N}_0} \iota_i(\mathcal{A}_0)$ in the weak operator topology. This implies the minimality of the random sequence.
An elementary computation shows $\mathcal{A}_I \simeq \mathbb{C}^{2^{|I|}}$ for any finite set $I \subset \mathbb{N}_0$. In the case of an infinite set I, we restrict the family of infinite product states $\{\psi(p)\}_{p\in(0,1)}$ to $\mathcal{A}_I$ and conclude again by the Kakutani theorem [Hid80] that these restricted states are mutually disjoint. This implies that the von Neumann algebra $\mathcal{A}_I$ contains a copy of $\mathcal{N}$ whenever $|I| = \infty$.
Now it is straightforward to verify the conditional full factorizability (CF)

$$E_{\mathcal{N}}(xy) = E_{\mathcal{N}}(x) E_{\mathcal{N}}(y)$$

for all $x \in \mathcal{A}_I$ and $y \in \mathcal{A}_J$ with disjoint subsets $I, J \subset \mathbb{N}_0$. Since all von Neumann algebras are commutative, it is immediate from the module property of conditional expectation that (CF) upgrades to (CI), i.e.

$$E_{\mathcal{N}}(xy) = E_{\mathcal{N}}(x)E_{\mathcal{N}}(y)$$

for all $x \in \mathcal{A}_I \vee \mathcal{N}$ and $y \in \mathcal{A}_J \vee \mathcal{N}$ with disjoint subsets $I, J \subset \mathbb{N}_0$. Thus the random sequence $(\iota_i)_{i \in \mathbb{N}_0}$ is full $\mathcal{N}$-independent. But $\mathcal{N} \not\subset \mathcal{A}_I \cap \mathcal{A}_J$ if one of the sets I or J is finite.

Remark 4.7. There are *-algebraic, C*-algebraic and W*-algebraic approaches to noncommutative probability and it is instructive to compare them at the hand of Example 4.6. Of course, the *-algebras $\mathcal{A}_I^{\mathrm{alg}} := \bigcup_{i \in I} \iota_i(\mathcal{A}_0)$ as well as its norm-closure are contained in the von Neumann algebra $\mathcal{A}_I$. The latter contains a copy of $L^\infty([0,1], \nu)$ if $|I|$ is infinite, but $\mathcal{A}_I^{\mathrm{alg}}$ and its norm closure do not.

5. Noncommutative i.i.d. sequences may be non-stationary

It is folklore in classical probability and free probability that independence resp. freeness of an identically distributed random sequence implies stationarity. But this implication fails in our broader context of noncommutative independence.

Theorem 5.1. *There exist full $\mathbb{C}$-independent identically distributed random sequences $\mathscr{I}$ which fail to be stationary.*

Proof, in particular of Theorem 0.2 (c) $\not\Leftarrow$ (d) and (c$_o$) $\not\Leftarrow$ (d$_o$). See Example 5.2 or Example 5.4 below. Since full $\mathbb{C}$-independence implies order $\mathbb{C}$-independence we have also shown $(\mathrm{c_o}) \not\Leftarrow (\mathrm{d_o})$. ☐

Let us first outline our strategy to produce such examples. Recall from the introduction that an infinite random sequence $\mathscr{I}$ with random variables

$$(\iota_n)_{n \geq 0} \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{M}, \psi)$$

is automatically identically distributed. Suppose now that $\mathscr{I}$ is stationary and $\mathbb{C}$-independent. Our goal is to 'perturbate' the random variables ι_n such that $\mathbb{C}$-independence is preserved, but stationary is obstructed. This can be done in two ways, for the domain or the codomain of each random variable ι_n.

Example 5.2 (Perturbation of codomain)**.** Consider $(\mathcal{R}, \mathrm{tr})$, the hyperfinite II_1-factor equipped with its normalized trace. Let (M_m, tr_m) be the complex $m \times m$-matrices equipped with the normalized trace. The canonical embeddings

$$M_2 \ni x \mapsto \iota_n(x) := \mathbb{1}_{M_2} \otimes \cdots \otimes \mathbb{1}_{M_2} \underset{n\text{-th position}}{\otimes \; x \; \otimes} \mathbb{1}_{M_2} \otimes \cdots$$

define the random sequence $\mathscr{I}$ with random variables

$$(\iota_n)_{n \geq 0} \colon (M_2, \mathrm{tr}_2) \to (\mathcal{R}, \mathrm{tr}).$$

It is easily verified that $\mathscr{I}$ is $\mathbb{C}$-independent and stationary. We will deform this random sequence to obtain a non-stationary random sequences as follows. Under the canonical identification of $M_2 \otimes M_2$ and M_4, the unitary matrix

$$U_\omega = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \omega \end{bmatrix}, \qquad |\omega| = 1,$$

defines the trace-preserving automorphism $x \mapsto U_\omega x U_\omega^*$ of $M_2 \otimes M_2$. It is well known in subfactor theory that the inclusions

$$\begin{array}{ccc} U_\omega(M_2 \otimes \mathbb{1}_{M_2})U_\omega^* & \subset & M_2 \otimes M_2 \\ \cup & & \cup \\ \mathbb{C}\,\mathbb{1}_{M_2\otimes M_2} & \subset & M_2 \otimes \mathbb{1}_{M_2} \end{array}$$

form a commuting square [Jon91, Rup95, JS97]. We canonically amplify this automorphism to the automorphism $\gamma_\omega \in \mathrm{Aut}(\mathcal{R}, \mathrm{tr})$ which acts trivial on all higher tensor product factors. Consider now the random sequence $\mathscr{I}^{(\omega)}$ with random variables $\big(\iota_n^{(\omega)}\big)_{n\geq 0}$ defined by

$$\iota_n^{(\omega)} := \begin{cases} \iota_n & \text{if } n \neq 1 \\ \gamma_\omega \iota_0 & \text{if } n = 1. \end{cases}$$

Note that $\mathscr{I}^{(1)}$ is the random sequence $\mathscr{I}$. Clearly $\mathscr{I}^{(\omega)}$ is identically distributed for any unimodular $\omega \in \mathbb{C}$. We note that the von Neumann algebras $\iota_n^{(\omega)}(M_2)$ mutually commute for $n \neq 1$. So do $\iota_1^{(\omega)}(M_2)$ and $\iota_n^{(\omega)}(M_2)$ for $n \geq 2$. We conclude from this that $\mathscr{I}^{(\omega)}$ is full $\mathbb{C}$-independent. But we calculate for $a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ that

$$\mathrm{tr}\left(\iota_0^{(\omega)}(a)\iota_1^{(\omega)}(a)\iota_0^{(\omega)}(a)\iota_1^{(\omega)}(a)\right) = \frac{1}{2}(\omega + \overline{\omega}),$$

and

$$\mathrm{tr}\left(\iota_2^{(\omega)}(a)\iota_3^{(\omega)}(a)\iota_2^{(\omega)}(a)\iota_3^{(\omega)}(a)\right) = 1.$$

This leads us to the conclusion that $\mathscr{I}^{(\omega)}$ is stationary if and only if $\omega = 1$.

Remark 5.3. Example 5.2 illustrates that the distribution of two $\mathbb{C}$-independent (identically distributed) random variables does not determine their joint distribution. This is in contrast to two distinguished examples for $\mathbb{C}$-independence, tensor independence and free independence. See [Spe97, BG02] for further information on the related universality properties.

We sketch next how local perturbations of random variables on their domain are capable to produce such effects. Suppose the minimal stationary random sequence $\mathscr{I}$ with random variables

$$(\iota_n)_{n\in\mathbb{N}_0} \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{M}, \psi)$$

is $\mathbb{C}$-independent (in the ordered or full sense). Furthermore, let $\gamma \equiv (\gamma_n)_{n\geq 0} \subset \mathrm{Aut}(\mathcal{A}_0, \varphi_0)$ be a sequence of 'local perturbations'. Then we can associate to each sequence γ a random sequence $\mathscr{I}^{(\gamma)}$ by putting

$$\iota_n^{(\gamma)} := \iota_n \circ \gamma_n.$$

The random sequence $\mathscr{I}^{(\gamma)}$ is again minimal and $\mathbb{C}$-independent. Suppose that there is a sequence γ with

$$(\iota_0, \iota_1, \ldots, \iota_{n-1}, \iota_n, \iota_{n+1}\ldots) \overset{\text{distr}}{\neq} (\iota_0^{(\gamma)}, \iota_1^{(\gamma)}, \ldots, \iota_{n-1}^{(\gamma)}, \iota_n, \iota_{n+1}\ldots)$$

for some $n \in \mathbb{N}$. We conclude immediately that the random sequence on the right hand side fails to be stationary, but it is still identically distributed and enjoys $\mathbb{C}$-independence.

Example 5.4 (Perturbation of domain)**.** Let $\ell^2(\mathbb{N})$ be the real Hilbert space of square-summable sequences and consider the q-Gaussian field $\Gamma_q(\ell^2(\mathbb{N}))$ for some fixed $0 < q < 1$. These fields are the von Neumann algebra generated by q-Gaussian field operators $\omega_q(f)$, $f \in \ell^2(\mathbb{N})$, acting on the q-deformed Fock space $\mathcal{F}_q(\ell^2(\mathbb{N}))$ (see [BKS97, GM02] for further details). $\Gamma_q(\ell^2(\mathbb{N}))$ is a non-hyperfinite II_1-factor and we denote its normalized trace by tr_q. The second quantization of the canonical unilateral shift on $\ell^2(\mathbb{N})$ provides us with a unital tr_q-preserving endomorphism α of $\Gamma_q(\ell^2(\mathbb{N}))$. Identify $\mathbb{R}$ with the subspace generated by the first coordinate of $\ell^2(\mathbb{N})$. Doing so we obtain the abelian von Neumann subalgebra $\Gamma_q(\mathbb{R}) \subset \Gamma_q(\ell^2(\mathbb{N}))$ and we denote the restriction of tr_q to this subalgebra by the same symbol. Now it is straightforward to see that

$$\iota_n := \alpha^n|_{\Gamma_q(\mathbb{R})}$$

defines a full $\mathbb{C}$-independent random sequence $\mathscr{I}$ with random variables

$$(\iota_{n\geq 0})\colon (\Gamma_q(\mathbb{R}), \operatorname{tr}_q) \to (\Gamma_q(\ell^2(\mathbb{N})), \operatorname{tr}_q),$$

which is of course stationary. Let $\gamma \in \operatorname{Aut}(\Gamma_q(\mathbb{R}), \operatorname{tr}_q)$ be fixed and consider the random sequence $\mathscr{I}_\gamma$ which is obtained from perturbating the first random variable of $\mathscr{I}$:

$$\iota_n^\gamma := \begin{cases} \iota_n & \text{if } n \neq 1 \\ \iota_0 \circ \gamma & \text{if } n = 0. \end{cases}$$

The central result by van Leeuwen and Maassen on the obstruction for q-deformation of the convolution product can be reformulated as:

Theorem 5.5 ([vLM96])**.** *Let $0 < q < 1$. There exists a 'perturbation' $\gamma \in \operatorname{Aut}(\Gamma_q(\mathbb{R}), \operatorname{tr}_q)$ such that*

$$\operatorname{tr}_q\left(\Big(\omega_q(f) + \alpha\big(\omega_q(f)\big)\Big)^4\right) \neq \operatorname{tr}_q\left(\Big(\gamma\big(\omega_q(f)\big) + \alpha\big(\omega_q(f)\big)\Big)^4\right)$$

for $0 \neq f \in \mathbb{R}$.

Note that $\omega(f), \gamma\big(\omega(f)\big)$ and $\alpha\big(\omega(f)\big)$ have identical distributions and each of the first two random variables is $\mathbb{C}$-independent from the third one. Thus the knowledge of the individual distributions of $\mathbb{C}$-independent random variables does not completely determine their joint distributions; this depends on the concrete realization of the random variables.

The 'perturbation' γ is of constructed in [vLM96] starting from a μ-preserving point transformation on the spectrum of the (selfadjoint) q-Gaussian field operator $\omega_q(f)$, for some fixed $f \in \mathbb{R}$, where μ is induced by the spectral measure of $\omega_q(f)$ with respect to tr_q.

Corollary 5.6. *$\mathscr{I}_\gamma$ is full $\mathbb{C}$-independent and non-stationary.*

Proof. It is immediate from its construction that $\mathscr{I}$ is full $\mathbb{C}$-independent. The perturbation γ of the domain of the first random variable does not effect its range. Thus $\mathscr{I}_\gamma$ is also full $\mathbb{C}$-independent.

Let $a := \omega_q(f)$ for notational convenience. A straightforward computation yields for the left hand side of the inequality in Theorem 5.5 that

$$\operatorname{tr}_q\left(\big(a + \alpha(a)\big)^4\right) = 2\operatorname{tr}_q(a^4) + 4\operatorname{tr}_q(a^2)\operatorname{tr}_q(a^2) + 2\operatorname{tr}_q\big(a\alpha(a)a\alpha(a)\big).$$

(Expand the product; use traciality, $\mathbb{C}$-independence, $\operatorname{tr}_q \circ \alpha = \operatorname{tr}_q$ and the centredness of a.) Similarly, the right hand side of this inequality simplifies to

$$\operatorname{tr}_q\Big(\big(\gamma(a)+\alpha(a)\big)^4\Big) = 2\operatorname{tr}_q(a^4) + 4\operatorname{tr}_q(a^2)\operatorname{tr}_q(a^2) + 2\operatorname{tr}_q\big(\gamma(a)\alpha(a)\gamma(a)\alpha(a)\big).$$

Since

$$\operatorname{tr}_q\big(\gamma(a)\alpha(a)\gamma(a)\alpha(a)\big) \neq \operatorname{tr}_q\big(a\alpha(a)a\alpha(a)\big)$$

by Theorem 5.5, we have

$$\operatorname{tr}_q\big(\gamma(a)\alpha(a)\gamma(a)\alpha(a)\big) \neq \operatorname{tr}_q\big(\alpha(a)\alpha^2(a)\alpha(a)\alpha^2(a)\big)$$

and consequently $(\iota_0 \circ \gamma, \iota_1, \iota_2, \ldots) \overset{\text{distr}}{\neq} (\iota_1, \iota_2, \iota_3, \ldots)$. □

The invariance of all finite joint distributions of an identically distributed random sequence under all local automorphisms seems to be a very strong condition. If the von Neumann algebra $\mathscr{M}$ is abelian and $\gamma \in \operatorname{Aut}(\mathcal{A}_0, \varphi_0)$ ergodic, such a local invariance property implies the $\mathbb{C}$-independence of the random sequence by an application of the mean ergodic theorem. In the noncommutative context, this observation invites to introduce 'top-order $\mathbb{C}$-independence' for a random sequence $\mathscr{I}$, i.e. the von Neumann algebras $\bigvee_{k<n} \iota_k(\mathcal{A}_0)$ and $\iota_n(\mathcal{A}_0)$ are $\mathbb{C}$-independent for all $n \in \mathbb{N}$. If $G \subset \operatorname{Aut}(\mathcal{A}_0, \varphi_0)$ is an amenable ergodic subgroup such that, for all $n \in \mathbb{N}$,

$$\psi\big(x\,\iota_n(a)\big) = \psi\big(x\,\iota_n(\gamma(a))\big)$$

for all $x \in \bigvee_{k<n} \iota_k(\mathcal{A}_0)$ and $\gamma \in G$, then the random sequence $(\iota_n)_{n\in\mathbb{N}_0}$ is already 'top-order $\mathbb{C}$-independent'.

Question 5.7. Suppose that a minimal random sequence $\mathscr{I}$ with random variables

$$(\iota_n)_{n\in\mathbb{N}_0}\colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{M}, \psi)$$

has joint distributions which are invariant under all 'local perturbations' $(\gamma_n)_{n\in\mathbb{N}} \subset \operatorname{Aut}(\mathcal{A}_0, \varphi_0)$:

$$(\iota_0, \iota_1, \iota_2, \ldots) \overset{\text{distr}}{=} (\iota_0 \circ \gamma_0, \iota_1 \circ \gamma_1, \iota_2 \circ \gamma_2, \ldots).$$

Does the ergodicity of $\operatorname{Aut}(\mathcal{A}_0, \varphi_0)$ imply that $\mathscr{I}$ is full $\mathbb{C}$-independent? And if so, can one show that this $\mathbb{C}$-independence must be either tensor independence or free independence?

6. Stationarity with strong mixing and noncommutative Bernoulli shifts

We provide a noncommutative generalization of the Kolmogorov Zero-One Law. Furthermore we show that conditional factorizability implies strong mixing in the context of stationarity. This leads us to a noncommutative generalization of classical Bernoulli shifts.

Theorem 6.1. *Let $\mathscr{I}$ be an order $\mathcal{N}$-factorizable random sequence where $\mathcal{N}$ is a ψ-conditioned von Neumann subalgebra of $\mathcal{M}^{\text{tail}}$. Then it holds $\mathcal{N} = \mathcal{M}^{\text{tail}}$. In particular, an order $\mathbb{C}$-independent random sequence has a trivial tail algebra.*

The last assertion is a noncommutative Kolmogorov Zero-One Law. Note also that order $\mathcal{N}$-factorizability (CF_o) is implied by (CI_o), (CF) or (CI).

Proof. Assume without loss of generality that $\mathscr{I}$ is minimal. We show first that $\mathcal{M}$ and $\mathcal{M}^{\mathrm{tail}}$ are $\mathcal{N}$-independent. Let $a \in \mathcal{M}^{\mathrm{tail}}$ and $x \in \mathcal{M}_{\mathbb{N}_0}^{\mathrm{alg}}$. Thus there exists some bounded subset $J \subset \mathbb{N}_0$ such that $x \in \mathcal{M}_J$. Because $\mathcal{M}^{\mathrm{tail}} \subset \mathcal{M}_{[n,\infty)}$ for all $n \in \mathbb{N}_0$, we can assume $J < [n, \infty)$. Consequently, the order $\mathcal{N}$-factorizability implies

$$E_{\mathcal{N}}(ax) = E_{\mathcal{N}}(a)E_{\mathcal{N}}(x).$$

Now let $x \in \mathcal{M}$. By minimality and Kaplansky's density theorem, there exists a bounded sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathcal{M}_{\mathbb{N}_0}^{\mathrm{alg}}$ of $\mathcal{M}$ such that $x = \textsc{wot-}\lim_k x_k$. Note that, for all k, we have $x_k \in \mathcal{M}_{J_k}$ with some bounded subset J_k. We conclude that, for any $y \in \mathcal{M}$,

$$\begin{aligned}\psi(yE_{\mathcal{N}}(ax)) &= \lim_k \psi(yE_{\mathcal{N}}(ax_k)) \\ &= \lim_k \psi(yE_{\mathcal{N}}(a)E_{\mathcal{N}}(x_k)) \\ &= \psi(yE_{\mathcal{N}}(a)E_{\mathcal{N}}(x)).\end{aligned}$$

This gives the factorization

$$E_{\mathcal{N}}(ax) = E_{\mathcal{N}}(a)E_{\mathcal{N}}(x) \tag{6.1}$$

for all $a \in \mathcal{M}^{\mathrm{tail}}$ and $x \in \mathcal{M}$. We claim that this factorization implies the $\mathcal{N}$-independence of $\mathcal{M}^{\mathrm{tail}}$ and $\mathcal{M}$. Indeed, the ψ-preserving conditional $E_{\mathcal{M}^{\mathrm{tail}}}$ from $\mathcal{M}$ onto $\mathcal{M}^{\mathrm{tail}}$ exist since $\mathcal{M}^{\mathrm{tail}}$ is globally σ_t^{ψ}-invariant. The latter is easily concluded from the fact that the ranges of the random variables ι_n are ψ-conditioned and the definition of $\mathcal{M}^{\mathrm{tail}}$. We are left to verify that (6.1) extends to elements $a \in \mathcal{M}^{\mathrm{tail}} \vee \mathcal{N}$ and $x \in \mathcal{M} \vee \mathcal{N}$. But this is evident, because $\mathcal{N} \subset \mathcal{M}$ and $\mathcal{N} \subset \mathcal{M}^{\mathrm{tail}}$. Thus $\mathcal{M}$ and $\mathcal{M}^{\mathrm{tail}}$ are $\mathcal{N}$-independent.

To prove $\mathcal{N} = \mathcal{M}^{\mathrm{tail}}$, we are left to show the inclusion $\mathcal{M}^{\mathrm{tail}} \subset \mathcal{N}$. We infer from the $\mathcal{N}$-independence of $\mathcal{M}$ and $\mathcal{M}^{\mathrm{tail}}$ that $\mathcal{M}^{\mathrm{tail}}$ and $\mathcal{M}^{\mathrm{tail}}$ are $\mathcal{N}$-independent. We use the module property of conditional expectations and $\mathcal{N}$-independence to get, for every $x \in \mathcal{M}^{\mathrm{tail}}$,

$$E_{\mathcal{N}}((x - E_{\mathcal{N}}(x))^*(x - E_{\mathcal{N}}(x)) = E_{\mathcal{N}}(x^*x) - E_{\mathcal{N}}(x^*)E_{\mathcal{N}}(x) = 0.$$

Now the faithfulness of $E_{\mathcal{N}}$ implies $x = E_{\mathcal{N}}(x)$ and thus $\mathcal{M}^{\mathrm{tail}} \subset \mathcal{N}$.

The last assertion is clear since order $\mathbb{C}$-factorizability and order $\mathbb{C}$-independence are equivalent (see Definition 4.1. □

Remark 6.2. The assumptions in Theorem 6.1 can be further weakened since an inspection of its proof shows that only the ranges of the random variables matter. It suffices that the probability space $(\mathcal{M}, \psi)$ is equipped with an order $\mathcal{N}$-factorizable family of ψ-conditioned von Neumann subalgebras $(\mathcal{M}_k)_{k\in\mathbb{N}}$.

It is well-known that the Kolmogorov Zero-One Law implies strong mixing properties of an independent stationary random sequence. Here we are interested in a conditioned noncommutative version of this classical result. It is convenient to formulate it in terms of the minimal stationary process $\mathscr{M}$ associated to a stationary random sequence $\mathscr{I}$.

Definition 6.3. A stationary process $\mathscr{M}$ or its endomorphism α is said to be *strongly mixing over* $\mathcal{N}$ if, for any $x \in \mathcal{M}$,

$$\textsc{wot-}\lim_{n\to\infty} \alpha^n(x) = E_{\mathcal{N}}(x).$$

Here $\mathcal{N}$ is a ψ-conditioned von Neumann subalgebra of $\mathcal{M}$.

Theorem 6.4. *Let the minimal stationary process $\mathscr{M}$ be order $\mathcal{N}$-factorizable for the ψ-conditioned subalgebra $\mathcal{N}$ of $\mathcal{M}^\alpha$. Then α is strongly mixing over $\mathcal{N}$. Moreover we have*

$$\mathcal{N} = \mathcal{M}^\alpha = \mathcal{M}^{\mathrm{tail}}.$$

In particular, these three subalgebras are trivial if $\mathscr{M}$ is order $\mathbb{C}$-independent.

The condition $\mathcal{N} \subset \mathcal{M}^\alpha$ is non-trivial if $\mathcal{M}^{\mathrm{tail}} \not\simeq \mathbb{C}$ (see Remark 6.5).

Proof. Since $\mathcal{M}^\alpha \subset \mathcal{M}^{\mathrm{tail}}$, we conclude $\mathcal{N} = \mathcal{M}^\alpha = \mathcal{M}^{\mathrm{tail}}$ from Theorem 6.1. We are left to prove the mixing properties. Suppose $x \in \mathcal{M}_I$ and $y \in \mathcal{M}_J$ for bounded sets $I, J \subset \mathbb{N}_0$. One calculates

$$\begin{aligned}
\lim_{n\to\infty} \psi(y^* \alpha^n(x)) &= \lim_{n\to\infty} \psi(E_{\mathcal{N}}(y^*\alpha^n(x))) \\
&= \lim_{n\to\infty} \psi(E_{\mathcal{N}}(y^*)E_{\mathcal{N}}(\alpha^n(x))) \\
&= \psi(E_{\mathcal{N}}(y^*)E_{\mathcal{N}}(x)) \\
&= \psi(y^* E_{\mathcal{N}}(x)).
\end{aligned}$$

Here we used that $J < (I + n)$ for n sufficiently large and applied order $\mathcal{N}$-factorizability to obtain the second equality. The third equality uses that $\mathcal{N} \subset \mathcal{M}^\alpha$ implies $E_{\mathcal{N}} \circ \alpha = E_{\mathcal{N}}$.

To extend these equations to arbitrary $x, y \in \mathcal{M}$, we use the minimality of the stationary process and approximate x and y by bounded sequences $(x_i)_i$ and, respectively, $(y_i)_i$ from the *-algebra $\mathcal{M}^{\mathrm{alg}}_{\mathbb{N}_0}$ in the strong operator topology. Since

$$\psi(y^*\alpha^n(x)) \quad = \quad \psi\big((y-y_i)^*\alpha^n(x)\big) + \psi\big(y_i^*\alpha^n(x-x_i)\big) + \psi\big(y_i^*\alpha^n(x_i)\big)$$

and since the estimates

$$\begin{aligned}
|\psi\big((y-y_i)^*\alpha^n(x)\big)| &\leq \psi(|y-y_i|^2)^{1/2}\psi(|x|^2)^{1/2}, \\
|\big(y_i^*\alpha^n(x-x_i)\big)| &\leq \psi(|y_i|^2)^{1/2}\psi(|x-x_i|^2)^{1/2}
\end{aligned}$$

are uniform in n, we conclude the convergence of $\psi(y^*\alpha^n(x))$ to $\psi(y^* E_{\mathcal{M}^{\mathrm{tail}}}(x))$ by an $\varepsilon/3$-argument. Now the claimed mixing property follows from the norm density of the functionals $\{\psi(y\,\cdot\,) \,|\, y \in \mathcal{M}\}$ in $\mathcal{M}_*$ and the boundedness of the set $\{\alpha^n(x) \,|\, n \in \mathbb{N}_0\}$. □

Remark 6.5. The condition $\mathcal{N} \subset \mathcal{M}^\alpha$ in Theorem 6.4 is non-trivial. Consider a minimal stationary process $\mathscr{M}$ with $\mathcal{N} = \mathcal{M}^{\mathrm{tail}} = \mathcal{M} \not\simeq \mathbb{C}$. Then $\mathscr{M}$ is $\mathcal{N}$-factorizable and $E_{\mathcal{N}}$ is the identity map on $\mathcal{M}$. Furthermore, α is easily seen to be an automorphism. It follows from Definition 6.3 that α is strongly mixing over $\mathcal{N}$ if and only if α is the identity.

Remark 6.6. Conditional order factorizability ($\mathrm{CF_o}$) is the weakest form of independence or factorizability introduced in Definition 4.1; thus Theorem 6.1 and Theorem 6.4 are also valid if ($\mathrm{CF_o}$) is replaced by (CF), ($\mathrm{CI_o}$) or (CI).

An important class of stationary processes in classical probability are Bernoulli shifts; and a noncommutative notion of such shifts emerges in [Küm88a] from the study of stationary quantum Markov processes. Here we are interested in their amalgamated version, as studied in [Rup95] and, in a bilateral continuous 'time' formulation, in [HKK04].

Definition 6.7. An *(ordered/full) Bernoulli shift (over $\mathcal{N}$)* is a minimal stationary process $\mathscr{B} = (\mathcal{B}, \chi, \beta, \mathcal{B}_0)$ with the following properties:

(i) $\mathcal{N} \subset \mathcal{B}^\alpha \cap \mathcal{B}_0$ is a χ-conditioned von Neumann subalgebra;

(ii) the canonical filtration $(\mathcal{B}_I)_{I \subset \mathbb{N}_0}$ is (order/full) $\mathcal{N}$-independent;

The endomorphism β is also called a *Bernoulli shift over $\mathcal{N}$ with generator* $\mathcal{B}_0$.

Note that this definition of a Bernoulli shift contains a subtle redundancy: one could drop the modular condition on the endomorphism β and conclude it from the fact that its ranges $\beta^n(\mathcal{B}_0)$ must be χ-conditioned, as required by our definition of independence. This entails that β commutes with σ_t^χ, the modular automorphism group of $(\mathcal{B}, \chi)$.

Corollary 6.8. *Let $\mathscr{M} = (\mathcal{M}, \psi, \alpha, \mathcal{M}_0)$ be a minimal stationary process. Further suppose $\mathcal{N} \subset \mathcal{M}^\alpha$ is a ψ-conditioned von Neumann subalgebra and $\mathscr{B} = (\mathcal{M}, \psi, \alpha, \mathcal{M}_0 \vee \mathcal{N})$. Then the following are equivalent:*

(a) *$\mathscr{M}$ is (order/full) $\mathcal{N}$-factorizable;*

(b) *$\mathscr{M}$ is (order/full) $\mathcal{N}$-independent;*

(c) *$\mathscr{B}$ is an (ordered/full) Bernoulli shift over $\mathcal{N}$.*

In particular, it holds $\mathcal{N} = \mathcal{M}^\alpha = \mathcal{M}^{\text{tail}}$.

Proof. We already know the equivalence of (a) and (b) from Theorem 4.2. The equivalence of (b) and (c) is also clear since the family $(\mathcal{M}_I)_{I \subset \mathbb{N}_0}$ is (order/full) $\mathcal{N}$-independent if and only if the family $(\mathcal{M}_I \vee \mathcal{N})_{I \subset \mathbb{N}_0}$ is so. We are left to show $\mathcal{N} = \mathcal{M}^\alpha = \mathcal{M}^{\text{tail}}$. But this is content of Theorem 6.4. □

We provide next a result which is useful for applications where one wants to identify a given process as a Bernoulli shift. Suppose $\mathscr{M} = (\mathcal{M}, \psi, \alpha, \mathcal{M}_0)$ is an (order/full) $\mathcal{N}$-factorizable minimal stationary process for some ψ-conditioned von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}^\alpha$. Furthermore let $\mathcal{C}_0$ be a ψ-conditioned von Neumann subalgebra of $\mathcal{M}_0$. Put

$$\mathcal{B} := \bigvee_{n \geq 0} \alpha^n(\mathcal{C}_0 \vee \mathcal{N}), \qquad \chi := \psi|_{\mathcal{B}}, \qquad \beta := \alpha|_{\mathcal{B}}, \qquad \mathcal{B}_0 := \mathcal{C}_0 \vee \mathcal{N}.$$

This defines the minimal stationary process $\mathscr{B} = (\mathcal{B}, \chi, \beta, \mathcal{B}_0)$ which is subject of the next result.

Corollary 6.9. *$\mathscr{B}$ is an (ordered/full) Bernoulli shift over $\mathcal{N}$ and $\mathcal{N} = \mathcal{B}^\beta = \mathcal{B}^{\text{tail}}$.*

Proof. Theorem 4.2 implies the (order/full) $\mathcal{N}$-independence of $\mathscr{M}$. Since $\mathcal{B}_0 \subset \mathcal{M}_0 \vee \mathcal{N}$, (order/full) $\mathcal{N}$-independence is inherited by the minimal stationary process $\mathscr{B}$. Now an application of Theorem 6.1 to the random sequence associated to $\mathscr{B}$ ensures $\mathcal{N} = \mathcal{B}^{\text{tail}}$. We are left to prove $\mathcal{N} \subset \mathcal{B}_0 \cap \mathcal{B}^\beta$. Clearly $\mathcal{N} \subset \mathcal{B}_0$. Thus it is suffices to show $\mathcal{N} = \mathcal{B}^\beta$. Since $\mathcal{N} = \mathcal{M}^\alpha$ by Theorem 6.4 and $\mathcal{N} \subset \mathcal{B}_0$, we have $\mathcal{M}^\alpha \subset \mathcal{B}_0$ and consequently $\mathcal{M}^\alpha \subset \mathcal{B}$. But this implies $\mathcal{B}^\beta = \mathcal{M}^\alpha$ and consequently $\mathcal{N} = \mathcal{B}^\beta$. □

Remark 6.10. Our notion of a Bernoulli shift is motivated from Kümmerer's work on noncommutative stationary Markov processes in [Küm85, Küm88b, Küm88a, Küm93, Küm96]. An ordered Bernoulli shift here is the unilateral discrete version of noncommutative continuous Bernoulli shifts introduced in [HKK04]. Note that Definition 6.7 of a Bernoulli shift is *not* restricted to tensor independence; it is casted in the broader context of conditional independence.

7. Spreadability implies conditional order independence

The main result of this section is Theorem 7.1 which is an integral part of the noncommutative extended de Finetti theorem, Theorem 0.2.

Theorem 7.1. *A spreadable random sequence $\mathscr{I}$ is stationary and order $\mathcal{M}^{\mathrm{tail}}$-independent.*

It is immediate from Definition 1.12 that spreadability implies the stationarity of a random sequence. Thus we can reformulate Theorem 7.1 in terms of stationary processes, as done in Theorem 7.2. Throughout this section, we consider the minimal stationary process

$$\mathscr{M} \equiv (\mathcal{M}, \psi, \alpha, \mathcal{M}_0)$$

and, replacing its generator $\mathcal{M}_0$ by $\mathcal{M}_0 \vee \mathcal{M}^\alpha$, the minimal stationary process

$$\mathscr{B} \equiv (\mathcal{M}, \psi, \alpha, \mathcal{M}_0 \vee \mathcal{M}^\alpha).$$

Theorem 7.2. *Suppose $\mathscr{M}$ is spreadable and minimal. Then $\mathscr{M}$ is order $\mathcal{M}^{\mathrm{tail}}$-independent and $\mathcal{M}^{\mathrm{tail}} = \mathcal{M}^\alpha$. In particular, $\mathscr{B}$ is an ordered Bernoulli shift.*

The proof of Theorem 7.2 needs some preparation and is postponed to the end of this section. It entails of course the proofs of Theorem 7.1 and Theorem 0.2 (b) ⇒($\mathrm{c_o}$) through the correspondence stated in Lemma 2.5.

Proposition 7.3. *Suppose the minimal stationary process $\mathscr{M}$ is spreadable. Then there exists the ψ-preserving conditional expectation $E_{\mathcal{M}^{\mathrm{tail}}}$ of $\mathcal{M}$ onto $\mathcal{M}^{\mathrm{tail}}$ and*

$$\textsc{wot-}\lim_n \alpha^n(x) = E_{\mathcal{M}^{\mathrm{tail}}}(x), \qquad x \in \mathcal{M}.$$

Moreover, we have $\mathcal{M}^{\mathrm{tail}} = \mathcal{M}^\alpha$.

Proof. Let $\mathcal{M}_I := \bigvee_{n\in I} \alpha^n(\mathcal{M}_0)$ for $I \subset \mathbb{N}_0$. Let $x, y \in \bigcup_{|I|<\infty} \mathcal{M}_I$. Consequently we can assume $x \in \mathcal{M}_I$ and $y \in \mathcal{M}_J$ such that there exists $N \in \mathbb{N}$ with $I \cap (J+N) = \emptyset$. We infer from spreadability that $\psi(y\alpha^n(x)) = \psi(y\alpha^{n+1}(x))$ for all $n \geq N$. Due to minimality this establishes the limit

$$\lim_{n\to\infty} \psi(y\alpha^n(x))$$

on the WOT-dense *-algebra $\bigcup_{|I|<\infty} \mathcal{M}_I$. A standard approximation argument ensures now the existence of this limit for $x, y \in \mathcal{M}$, using the norm density of the functionals $\{\psi(y\cdot) \mid y \in \mathcal{M}\}$ and the boundedness of the set $\{\alpha^n(x) \mid n \in \mathbb{N}\}$. We conclude from this that the pointwise WOT-limit of the sequence $(\alpha^n)_n$ defines a linear map $Q\colon \mathcal{M} \to \mathcal{M}$ such that $Q(\mathcal{M}) \subset \mathcal{M}^{\mathrm{tail}}$.

It is easily seen that the linear map Q enjoys

$$\psi = \psi \circ Q \quad \text{and} \quad \|Q(x)\| \leq \|x\| \text{ for } x \in \mathcal{M}.$$

Thus Q is a conditional expectation from $\mathcal{M}$ onto $\mathcal{M}^{\mathrm{tail}}$, if we can insure that $Q(x) = x$ for all $x \in \mathcal{M}^{\mathrm{tail}}$. To this end let $x \in \mathcal{M}^{\mathrm{tail}}$ and $y \in \bigcup_{|I|<\infty} \mathcal{M}_I$. We infer from $\mathcal{M}^{\mathrm{tail}} \subset \alpha^N(\mathcal{M})$ and $\mathcal{M}_{[N,\infty)} \subset \alpha^N(\mathcal{M})$ for all $N \in \mathbb{N}$ that there exists some $N \in \mathbb{N}$ such that $x \in \alpha^N(\mathcal{M})$ and $y \in \mathcal{M}_{[0,N-1]}$. We approximate $x \in \mathcal{M}$ in the WOT-sense by a sequence $(x_k)_k \subset \bigcup_{|I|<\infty} \alpha^N(\mathcal{M}_I)$ and conclude further from the definition of Q and from spreadability that

$$\begin{aligned}\psi(yQ(x)) &= \lim_k \psi(yQ(x_k)) = \lim_k \lim_n \psi(y\alpha^n(x_k)) \\ &= \lim_k \psi(yx_k) = \psi(yx).\end{aligned}$$

This shows that $Q(x) = x$ for all $x \in \mathcal{M}^{\text{tail}}$. Thus Q is the conditional expectation of $\mathcal{M}$ onto $\mathcal{M}^{\text{tail}}$ with respect to ψ (see [Tak03, Chapter IX, Definition 4.1]), which we denote from now on by $E_{\mathcal{M}^{\text{tail}}}$.

We need to identify the tail algebra as the fixed point algebra. Proposition 7.3 gives pointwise $E_{\mathcal{M}^{\text{tail}}}E_{\mathcal{M}^\alpha} = \text{WOT-}\lim_n \alpha^n E_{\mathcal{M}^\alpha} = E_{\mathcal{M}^\alpha}$ and thus $\mathcal{M}^\alpha \subset \mathcal{M}^{\text{tail}}$. The inclusion $\mathcal{M}^{\text{tail}} \subset \mathcal{M}^\alpha$ follows from $\alpha E_{\mathcal{M}^{\text{tail}}} = \lim_n \alpha\,\alpha^n = E_{\mathcal{M}^{\text{tail}}}$ in the pointwise WOT-sense. □

Remark 7.4. The proof of Proposition 7.3 shows that the ψ-preserving conditional expectation onto the tail algebra $\mathcal{M}^{\text{tail}}$ and the fixed point algebra $\mathcal{M}^\alpha$ of the endomorphism α exist under weaker assertions. One does not need that α and the modular automorphism group σ_t^ψ commute (this compatibility condition is required in Definition 2.1).

It is convenient to use Speicher's notion of multilinear maps also for the endomorphism α. we put

$$\alpha[\mathbf{i};\mathbf{a}] := \alpha^{\mathbf{i}(1)}(a_1)\alpha^{\mathbf{i}(2)}(a_2)\cdots\alpha^{\mathbf{i}(n)}(a_n)$$

for n-tuples $\mathbf{i}\colon [n] \to \mathbb{N}_0$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathcal{M}_0$.

Definition 7.5. A stationary process $\mathscr{M} = (\mathcal{M}, \psi, \alpha, \mathcal{M}_0)$ or its endomorphism α is $\mathcal{N}$*-spreadable* if there exists a ψ-conditioned von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$ such that

$$E_{\mathcal{N}}(\alpha[\mathbf{i};\mathbf{a}]) = E_{\mathcal{N}}(\alpha[\mathbf{j};\mathbf{a}])$$

for any $n \in \mathbb{N}$, $\mathbf{i},\mathbf{j}\colon [n] \to \mathbb{N}_0$ with $\mathbf{i} \sim_o \mathbf{j}$ and $\mathbf{a} \in \mathcal{M}_0^n$.

Lemma 7.6. *The following are equivalent for a minimal stationary process* $\mathscr{M}$*:*

(a) $\mathscr{M}$ *is spreadable;*
(b) $\mathscr{M}$ *is* $\mathcal{M}^{\text{tail}}$*-spreadable;*
(c) $\mathscr{M}$ *is* $\mathcal{M}^\alpha$*-spreadable.*

Proof. (b) and (c) are equivalent since $\mathcal{M}^{\text{tail}} = \mathcal{M}^\alpha$ by Proposition 7.3. Obviously (b) implies (a) and we are left to prove the converse. Let us first treat the case $\mathcal{M}^{\text{tail}} \subset \mathcal{M}_0$. We already know $\mathcal{M}^{\text{tail}} = \mathcal{M}^\alpha$ from Proposition 7.3. Consider the n-tuple $(ax_1, x_2, \ldots, x_n) \in \mathcal{M}_0^n$ with $a \in \mathcal{M}^\alpha$. We conclude from this that, for $\mathbf{i},\mathbf{j}\colon [n] \to \mathbb{N}_0$ with $\mathbf{i} \sim_o \mathbf{j}$,

$$\begin{aligned}\psi\big(a\alpha[\mathbf{i};x_1,x_2,\ldots,x_n]\big) &= \psi\big(\alpha[\mathbf{i};ax_1,x_2,\ldots,x_n]\big) = \psi\big(\alpha[\mathbf{j};ax_1,x_2,\ldots,x_n]\big) \\ &= \psi\big(a\alpha[\mathbf{j};x_1,x_2,\ldots,x_n]\big).\end{aligned}$$

Using $\psi = \psi \circ E_{\mathcal{M}^{\text{tail}}}$ and the module property of $E_{\mathcal{M}^{\text{tail}}}$, we conclude that α is conditionally $\mathcal{M}^{\text{tail}}$-spreadable by standard arguments.

The more general case $\mathcal{M}^{\text{tail}} \not\subset \mathcal{M}_0$ is treated similar. We approximate $a \in \mathcal{M}^{\text{tail}}$ by a sequence $(a_k)_{k\geq 0} \subset \mathcal{M}$ such that

$$a_k \in \bigcup_{l\geq k} \alpha^l(\mathcal{M}_0) \qquad \text{and} \qquad a = \text{SOT-}\lim_{k\to\infty} a_k.$$

Thus we can assume that each a_k is a linear combination of monomials $\alpha[\mathbf{i}_k;\mathbf{a}_k]$, for some n_k-tuple $\mathbf{i}_k\colon [n_k] \to \{k, k+1, \ldots\}$ and $\mathbf{a} \in \mathcal{M}_0^{n_k}$. Now we compute as before that, for $\mathbf{i},\mathbf{j}\colon [n] \to \mathbb{N}_0$ with $\mathbf{i} \sim_o \mathbf{j}$ and sufficiently large k,

$$\psi\big(\alpha[\mathbf{i}_k;\mathbf{a}_k]\alpha[\mathbf{i}; x_1, x_2, \ldots, x_n]\big) \quad = \quad \psi\big(\alpha[\mathbf{i}_k;\mathbf{a}_k]\alpha[\mathbf{j}; x_1, x_2, \ldots, x_n]\big).$$

This equality extends by linearity and weak* density arguments to

$$\psi\big(a\alpha[\mathbf{i}; x_1, x_2, \ldots, x_n]\big) \quad = \quad \psi\big(a\alpha[\mathbf{j}; x_1, x_2, \ldots, x_n]\big)$$

for every $a \in \mathcal{M}^{\mathrm{tail}}$. We conclude from this the $\mathcal{M}^{\mathrm{tail}}$-spreadability of the stationary process. □

Lemma 7.7. *Suppose $\mathscr{M}$ be a minimal stationary process. If $\mathscr{M}$ is spreadable, then $\mathscr{M}$ is order $\mathcal{M}^{\mathrm{tail}}$-factorizable.*

Proof. We need to show that the canonical filtration $(\mathcal{M}_I)_{I\subset\mathbb{N}_0}$ satisfies the factorization rule

$$E_{\mathcal{M}^{\mathrm{tail}}}(xy) = E_{\mathcal{M}^{\mathrm{tail}}}(x)E_{\mathcal{M}^{\mathrm{tail}}}(y)$$

for all $x \in \mathcal{M}_I$ and $y \in \mathcal{M}_J$ whenever $I < J$ or $I > J$. Let $x \in \mathcal{M}_I^{\mathrm{alg}}$ and $y \in \mathcal{M}_J^{\mathrm{alg}}$. Then, for all $n \in \mathbb{N}_0$,

$$E_{\mathcal{M}^{\mathrm{tail}}}(xy) = E_{\mathcal{M}^{\mathrm{tail}}}(x\alpha^n(y)),$$

since spreadability implies $\mathcal{M}^{\mathrm{tail}}$-spreadability (Lemma 7.6). We use the mixing properties of α (Proposition 7.3) to conclude

$$E_{\mathcal{M}^{\mathrm{tail}}}(xy) = \textsc{wot-}\lim_{n\to\infty} E_{\mathcal{M}^{\mathrm{tail}}}(x\alpha^n(y)) = E_{\mathcal{M}^{\mathrm{tail}}}(x)E_{\mathcal{M}^{\mathrm{tail}}}(y).$$

This establishes the order $\mathcal{M}^{\mathrm{tail}}$-factorizability of a spreadable stationary process. □

Proof of Theorem 7.2. Lemma 7.7 shows that $\mathscr{M}$ is order $\mathcal{M}^{\mathrm{tail}}$-factorizable and Proposition 7.3 insures $\mathcal{M}^{\mathrm{tail}} = \mathcal{M}^{\alpha}$. Thus Theorem 4.2 applies for $\mathcal{N} = \mathcal{M}^{\mathrm{tail}}$ and ensures that $\mathcal{M}$ is conditionally $\mathcal{M}^{\mathrm{tail}}$-independent. Finally, Corollary 6.8 entails that $\mathscr{B}$ is an ordered Bernoulli shift over $\mathcal{M}^{\mathrm{tail}}$. □

8. Spreadability implies conditional full independence

We have already shown in the previous section that spreadability implies conditional order independence. Here this result will be strengthened to conditional full independence.

Theorem 8.1. *A spreadable random sequence $\mathscr{I}$ is stationary and full $\mathcal{M}^{\mathrm{tail}}$-independent.*

Theorem 8.1 establishes the implication (b) ⇒ (c) of Theorem 0.2, the noncommutative extended de Finetti theorem. We will prove it in terms of the corresponding stationary process $\mathscr{M} = (\mathcal{M}, \psi, \alpha, \mathcal{M}_0)$ and, replacing the generator $\mathcal{M}_0$ by $\mathcal{M}_0 \vee \mathcal{M}^{\alpha}$, denote by $\mathscr{B}$ the stationary process $(\mathcal{M}, \psi, \alpha, \mathcal{M}_0 \vee \mathcal{M}^{\alpha})$.

Theorem 8.2. *Suppose $\mathscr{M}$ is spreadable and minimal. Then $\mathscr{M}$ is full $\mathcal{M}^{\mathrm{tail}}$-independent and $\mathcal{M}^{\mathrm{tail}} = \mathcal{M}^{\alpha}$. In particular, $\mathscr{B}$ is a full Bernoulli shift.*

The proofs of Theorem 8.1 and Theorem 8.2 require a certain refined version of the mean ergodic theorem. Let us start with its usual formulation and include for the convenience of the reader how its proof reduces to the usual result for contractions on Hilbert spaces.

Theorem 8.3. *Let $(\mathscr{M}, \psi)$ be a probability space and α a ψ-preserving endomorphism of $\mathcal{M}$. Then we have, for each $x \in \mathcal{M}$,*

$$\text{SOT-}\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \alpha^k(x) = E_{\mathcal{M}^\alpha}(x).$$

Proof. The strong operator topology and the ψ-topology generated by the maps $x \mapsto \psi(x^*x)^{1/2}$, $x \in \mathcal{M}$, coincide on norm bounded sets in $\mathcal{M}$. Thus this mean ergodic theorem is an immediate consequence of the usual mean ergodic theorem in Hilbert spaces (see [Pet83, Theorem 1.2] for example). □

This mean ergodic theorem would allow us to give an alternative proof of that spreadability implies conditional order independence ($\mathrm{CI_o}$), after having identified the tail algebra as the fixed point algebra of the stationary process in Proposition 7.3 and established conditional spreadability in Lemma 7.6.

We illustrate this by an example. Given the stationary process $(\mathcal{M}, \psi, \alpha, \mathcal{M}_0)$, let $a, b \in \mathcal{M}_0$ and consider

$$\begin{aligned}\mathcal{M}_{\{1,2\}} \ni x &= \alpha(a)\alpha^2(a)\alpha(a)\alpha^2(a),\\ \mathcal{M}_{\{3,4\}} \ni y &= \alpha^4(b)\alpha^3(b)\alpha^4(b)\alpha^3(b)\alpha^4(b).\end{aligned}$$

We have $\{1,2\} < \{3,4\}$ and thus spreadability implies

$$E_{\mathcal{M}^\alpha}(xy) = E_{\mathcal{M}^\alpha}\big(x\alpha^n(y)\big) = E_{\mathcal{M}^\alpha}\Big(x\frac{1}{n}\sum_{k=0}^{n-1}\alpha^k(y)\Big)$$

for all $n \geq 1$. Thus Theorem 8.3 implies $E_{\mathcal{M}^\alpha}(xy) = E_{\mathcal{M}^\alpha}(x)E_{\mathcal{M}^\alpha}(y)$.

But such an argument falls short of establishing the apparently stronger version, conditional full independence (CI). For example, consider the two elements

$$\begin{aligned}x &= \alpha(a)\alpha^3(a)\alpha(a)\alpha^3(a),\\ y &= \alpha^4(b)\alpha^2(b)\alpha^4(b)\alpha^2(b)\alpha^4(b),\end{aligned}$$

Thus we have $x \in \mathcal{M}_I$ and $y \in \mathcal{M}_J$ with $I = \{1,3\}$ and $J \in \{2,4\}$. Since the tuples $(1,3,1,3,4,2,4,2,4)$ and $(1,3,1,3,4+n,2+n,4+n,2+n,4+n)$ are order equivalent if and only if $n = 0$, the previous arguments fails. We observe that spreadability implies, in particular,

$$\begin{aligned}E_{\mathcal{M}^\alpha}(xy) &= E_{\mathcal{M}^\alpha}\big(x\alpha^{4+n}(b)\alpha^2(b)\alpha^{4+n}(b)\alpha^2(b)\alpha^{4+n}(b)\big)\\ &= E_{\mathcal{M}^\alpha}\Big(x\frac{1}{n}\sum_{k=0}^{n-1}\alpha^{4+k}(b)\alpha^2(b)\alpha^{4+k}(b)\alpha^2(b)\alpha^{4+k}(b)\Big).\end{aligned}$$

but a direct application of the mean ergodic theorem is still out of reach.

To overcome such difficulties we need to provide a more elaborated version of Theorem 8.3 which allows us to preserve relative localisation properties of the canonical filtration $(\mathcal{M}_I)_I$ while performing mean ergodic averages. Since this result is of interest in its own, we formulate it in greater generality as necessary for our purposes.

Theorem 8.4. *Let $(\mathcal{M}, \psi)$ be a probability space and suppose $\{\alpha_N\}_{N\in\mathbb{N}_0}$ is a family of ψ-preserving completely positive linear maps of $\mathcal{M}$ satisfying*

(i) $\mathcal{M}^{\alpha_N} \subset \mathcal{M}^{\alpha_{N+1}}$ *for all* $N \in \mathbb{N}_0$*;*
(ii) $\mathcal{M} = \bigvee_{N\in\mathbb{N}_0} \mathcal{M}^{\alpha_N}$.

Furthermore let

$$M_N^{(n)} := \frac{1}{n}\sum_{k=0}^{n-1}\alpha_N^k \quad \textit{and} \quad T_N := \overrightarrow{\prod}_{l=0}^{N}\alpha_l^{lN}M_l^{(N)}.$$

Then we have

$$\textsc{sot-}\lim_{N\to\infty} T_N(x) = E_{\mathcal{M}^{\alpha_0}}(x)$$

for any $x \in \mathcal{M}$*.*

Proof. Since the family $\{T_N \mid N \in \mathbb{N}_0\}$ is bounded, its pointwise SOT-convergence follows by a standard approximation argument if we can establish this convergence on the weak*-dense *-subalgebra $\bigcup_{N\in\mathbb{N}_0}\mathcal{M}^{\alpha_N}$ of $\mathcal{M}$.

Let $x \in \mathcal{M}^{\alpha_{N_0}}$ for some $N_0 \in \mathbb{N}$ and $N \geq N_0$. Since $\alpha_N(x) = x$ and thus $M_N^{(n)}(x) = x$, the ordered product has at most N_0 non-trivially acting factors:

$$T_N(x) = \Big(\overrightarrow{\prod}_{l=0}^{N}\alpha_l^{lN}M_l^{(N)}\Big)(x) = \Big(\overrightarrow{\prod}_{l=0}^{N_0-1}\alpha_l^{lN}M_l^{(N)}\Big)(x).$$

The assertions on the fixed point algebras $\mathcal{M}^{\alpha_k}$ imply that, for any $k \leq N$ and $n \in \mathbb{N}$,

$$E_{\mathcal{M}^{\alpha_k}}\alpha_N = E_{\mathcal{M}^{\alpha_k}} \quad \text{and} \quad E_{\mathcal{M}^{\alpha_k}}M_N^{(n)} = E_{\mathcal{M}^{\alpha_k}}.$$

Thus we can rewrite $T_N(x)$ as a finite telescope sum, assuming $N \geq N_0$:

$$\begin{aligned}
T_N(x) &= M_0^{(N)}\alpha_1^N M_1^{(N)}\alpha_2^{2N}M_2^{(N)}\cdots\alpha_N^{N^2}M_N^{(N)}(x)\\
&= M_0^{(N)}\alpha_1^N M_1^{(N)}\alpha_2^{2N}M_2^{(N)}\cdots\alpha_{N_0-1}^{(N_0-1)N}M_{N_0-1}^{(N)}(x)\\
&= \Big(\overrightarrow{\prod}_{l=0}^{N_0-1}\alpha_l^{lN}(M_l^{(N)} - E_{\mathcal{M}^{\alpha_l}})\Big)E_{\mathcal{M}^{\alpha_{N_0}}}(x)\\
&\quad + \Big(\overrightarrow{\prod}_{l=0}^{N_0-2}\alpha_l^{lN}(M_l^{(N)} - E_{\mathcal{M}^{\alpha_l}})\Big)E_{\mathcal{M}^{\alpha_{N_0-1}}}(x)\\
&\quad + \Big(\overrightarrow{\prod}_{l=0}^{N_0-3}\alpha_l^{lN}(M_l^{(N)} - E_{\mathcal{M}^{\alpha_l}})\Big)E_{\mathcal{M}^{\alpha_{N_0-2}}}(x)\\
&\quad + \cdots\\
&\quad + (M_0^{(N)} - E_{\mathcal{M}^{\alpha_0}})\alpha_1^N(M_1^{(N)} - E_{\mathcal{M}^{\alpha_1}})E_{\mathcal{M}^{\alpha_2}}(x)\\
&\quad + (M_0^{(N)} - E_{\mathcal{M}^{\alpha_0}})E_{\mathcal{M}^{\alpha_1}}(x)\\
&\quad + E_{\mathcal{M}^{\alpha_0}}(x).
\end{aligned}$$

The strong operator topology and the ψ-topology generated by $x \mapsto \|x\|_\psi^2 := \psi(x^*x)$ coincide on bounded sets of $\mathcal{M}$. Thus

$$\Big\|\Big(\overrightarrow{\prod}_{l=0}^{k-1}\alpha_l^{lN}(M_l^{(N)} - E_{\mathcal{M}^{\alpha_l}})\Big)E_{\mathcal{M}^{\alpha_k}}(x)\Big\|_\psi \leq 2^{k-1}\Big\|(M_{k-1}^{(N)} - E_{\mathcal{M}^{\alpha_{k-1}}})E_{\mathcal{M}^{\alpha_k}}(x)\Big\|_\psi,$$

and the usual mean ergodic theorem, Theorem 8.3, entail that all terms of above telescope sum, except $E_{\mathcal{M}^{\alpha_0}}(x)$, vanish in the limit $N \to \infty$. □

We will connect this refined mean ergodic theorem to partial shifts which canonically emerge from a spreadable endomorphism. Recall for this purpose the notion of partial shifts θ_N of $\mathbb{N}_0$ and their relation to order invariance of tuples (see Remark 1.9):

$$\theta_N(n) = \begin{cases} n & \text{if } n < N;\\ n+1 & \text{if } n \geq N.\end{cases}$$

Clearly θ_N is an order preserving map of $\mathbb{N}_0$ into itself and so are the compositions of such maps with $N \in \mathbb{N}_0$. Here we are interested in compositions of the type

$$\theta_{N,\vec{l}_N} := \overrightarrow{\prod}_{i=0}^{N} \theta_i^{iN+l_i} = \theta_0^{l_0}\theta_1^{N+l_1}\theta_2^{2N+l_2}\cdots\theta_{N-1}^{(N-1)N+l_{N-1}}\theta_N^{N^2+l_N},$$

where

$$\vec{l}_N = (l_0, l_1, \ldots, l_N) \in \{0, 1, \ldots, N-1\}^{N+1}.$$

Note that the θ_i's in the ordered product do not commute for different i's. We record two simple, but crucial properties of this composition.

Lemma 8.5. *For any $(N+1)$-tuples $\vec{l}_N, \vec{k}_N \in \{0, 1, \ldots, N-1\}^{N+1}$, it holds*

$$\theta_{N,\vec{l}_N}(i) < \theta_{N,\vec{k}_N}(j) \quad \text{whenever } i < j < N, \text{ and}$$
$$\theta_{N,\vec{l}_N}(I) \cap \theta_{N,\vec{k}_N}(J) = \emptyset \quad \text{whenever } I \cap J = \emptyset \text{ and } \max I \cup J < N.$$

Proof. Since all θ_N's are order preserving, it suffices to consider $j = i+1$. One calculates

$$\theta_{N,\vec{l}_N}(i+1) - \theta_{N,\vec{l}_N}(i) = 1 + \sum_{j=0}^{i}(k_j - l_j) + \big((i+1)N + k_{i+1}\big) > 0.$$

Moreover this ensures that the images of disjoint sets I, J (bounded by N) are disjoint. □

Suppose for the remainder of this section that the stationary process $\mathscr{M} = (\mathcal{M}, \psi, \alpha, \mathcal{M}_0)$ is minimal and let, for $N \in \mathbb{N}$,

$$\mathcal{M}_{N-1} := \bigvee_{0 \le k < N} \alpha^k(\mathcal{M}_0).$$

Spreadability of $\mathscr{M}$ allows us to promote the partial shifts θ_N of $\mathbb{N}_0$ to endomorphisms of $\mathcal{M}$. Let

$$\alpha[\mathbf{i}; \mathbf{a}] := \alpha^{\mathbf{i}(1)}(a_1)\alpha^{\mathbf{i}(2)}(a_2)\cdots\alpha^{\mathbf{i}(n)}(a_n)$$

for n-tuples $\mathbf{i}\colon [n] \to \mathbb{N}_0$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathcal{M}_0^n$.

Lemma 8.6. *Suppose the endomorphism α of $\mathcal{M}$ is spreadable and let $N \in \mathbb{N}_0$. Then the complex linear extension of the map*

$$\alpha[\mathbf{i}; \mathbf{a}] \mapsto \alpha[\theta_N \circ \mathbf{i}; \mathbf{a}]$$

defines a ψ-preserving unital endomorphism α_N of $\mathcal{M}$, such that

$$\mathcal{M}_N \subset \mathcal{M}^{\alpha_{N+1}}.$$

In particular, $\mathscr{M}_N := (\mathcal{M}, \psi, \alpha_N, \mathcal{M}_N)$ is a minimal stationary process.

Proof. The map α_N is well-defined on the *-algebra $\mathcal{M}_{\mathbb{N}_0}^{\mathrm{alg}}$, the $\mathbb{C}$-linear span of monomials $\alpha[\mathbf{i}; \mathbf{a}]$. Indeed, the faithfulness of ψ and spreadability give

$$\sum_k \alpha[\theta_N \circ \mathbf{i}_k; \mathbf{a}_k] = 0 \quad \Leftrightarrow \quad \psi\Big(\big|\sum_k \alpha[\theta_N \circ \mathbf{i}_k; \mathbf{a}_k]\big|^2\Big) = \psi\Big(\big|\sum_k \alpha[\mathbf{i}_k; \mathbf{a}_k]\big|^2\Big) = 0$$
$$\Leftrightarrow \quad \sum_k \alpha[\mathbf{i}_k; \mathbf{a}_k] = 0.$$

Thus α_N is well-defined on $\mathcal{M}_{\mathbb{N}_0}^{\mathrm{alg}}$. Now it is routine to check that α_N extends to a ψ-preserving unital endomorphism of $\mathcal{M}$, denoted by the same symbol. The inclusion

$\mathcal{M}_{N-1} \subset \mathcal{M}^{\alpha_N}$ is immediately concluded by approximation from the definition of α_N on $\mathcal{M}_{\mathbb{N}_0}^{\mathrm{alg}}$. It is also clear that α_N commutes with the modular automorphism group of $(\mathcal{M}, \psi)$ since α does so. Thus $(\mathcal{M}, \psi, \alpha_N, \mathcal{M}_N)$ is a stationary process which is easily seen to be minimal. □

Corollary 8.7. *The minimal stationary processes $\mathscr{M}_N$ and their endomorphisms α_N are spreadable. Moreover, it holds for $N \in \mathbb{N}_0$:*

(i) $\alpha_{N+1}|_{\alpha_N(\mathcal{M})} = \alpha_N|_{\alpha_N(\mathcal{M})}$;
(ii) $\mathcal{M}^{\alpha_N} \subset \mathcal{M}^{\alpha_{N+1}}$;
(iii) $\mathcal{M} = \bigvee_{N\in\mathbb{N}_0} \mathcal{M}^{\alpha_N}$.

Proof. The spreadability of $\mathscr{M}_N$ is immediate from definition of α_N in Lemma 8.6 and the spreadability of α.

(i) Clearly, $\theta_{N+1}|_{\theta_N(\mathbb{N}_0)} = \theta_N|_{\theta_N(\mathbb{N}_0)}$. Thus α_{N+1} and α_N coincide on the $\mathbb{C}$-linear span of all monomials of the form $\alpha[\theta_N \circ \mathbf{i}; \mathbf{a}] = \alpha_N\Big(\alpha[\mathbf{i}; \mathbf{a})]\Big)$. Now the assertion follows from the weak*-density of this span in $\alpha_N(\mathcal{M})$.

(ii) $\mathcal{M}^{\alpha_N}$ is contained in $\alpha_N(\mathcal{M})$. By (i), α_N and α_{N+1} coincide on $\alpha_N(\mathcal{M})$. Thus $\mathcal{M}^{\alpha_N} \subset \mathcal{M}^{\alpha_{N+1}}$.

(iii) This is evident from the minimality of $\mathscr{M}$ since $\bigvee_{0\le n<N} \alpha^n(\mathcal{M}_0) \subset \mathcal{M}^{\alpha_N}$ by Lemma 8.6. □

Remark 8.8. We do not know at the time of this writing if the fixed point algebras $\mathcal{M}^{\alpha_N}$ can be identified as $\mathcal{M}^{\alpha_N} = \bigvee_{0\le n<N} \alpha^n(\mathcal{M}_0) \vee \mathcal{M}^{\alpha}$.

Proof of Theorem 8.2. We need to show that

$$E_{\mathcal{M}^\alpha}(xy) = E_{\mathcal{M}^\alpha}(x) E_{\mathcal{M}^\alpha}(y)$$

for all $x \in \mathcal{M}_I$ and $y \in \mathcal{M}_J$ with $I \cap J = \emptyset$. We start with disjoint finite sets I and J, and elements of the form

$$x = \alpha[\mathbf{i}; \mathbf{a}] \quad \text{and} \quad y = \alpha[\mathbf{j}; \mathbf{b}],$$

for p-tuples $\mathbf{i}\colon [p] \to I$, $\mathbf{a} \in \mathcal{M}_0^p$ and q-tuples $\mathbf{j}\colon [q] \to J$, $\mathbf{b} \in \mathcal{M}_0^q$.

Recall that $\mathscr{M}_\alpha$ is $\mathcal{M}^\alpha$-spreadable by Lemma 7.6 and so

$$\begin{aligned} E_{\mathcal{M}^{\alpha_0}}(xy) &= E_{\mathcal{M}^\alpha}\Big(\alpha[\mathbf{i}; \mathbf{a}]\, \alpha[\mathbf{j}; \mathbf{b}]\Big) \\ &= E_{\mathcal{M}^\alpha}\Big(\alpha[\theta_{N,\vec{k}_N} \circ \mathbf{i}; \mathbf{a}]\, \alpha[\theta_{N,\vec{k}_N} \circ \mathbf{j}, \mathbf{b}]\Big) \end{aligned}$$

for any $\vec{k}_N \in \{0, 1, \ldots, N-1\}^{N+1}$ and $N > \max I \cup J$. By Lemma 8.5, the maps $\theta_{N,\vec{k}_N}$ are order preserving on $\mathbb{N}_0$ and $I \cap J = \emptyset$ implies $\theta_{N,\vec{k}_N}(I) \cap \theta_{N,\vec{l}_N}(J) = \emptyset$ for any $(N+1)$-tuples $\vec{k}_N, \vec{l}_N \in \{0, 1, \ldots, N-1\}^{N+1}$. Thus

$$E_{\mathcal{M}^\alpha}\Big(\alpha[\mathbf{i}; \mathbf{a}]\, \alpha[\mathbf{j}; \mathbf{b}]\Big) = E_{\mathcal{M}^\alpha}\Big(\alpha[\theta_{N,\vec{k}_N} \circ \mathbf{i}; \mathbf{a}]\, \alpha[\theta_{N,\vec{l}_N} \circ \mathbf{j}, \mathbf{b}]\Big)$$

for all $\vec{k}_N, \vec{l}_N$. Consequently we can pass on the right side of this equation to the mean ergodic averages by summing over the variables $k_0, k_1, \ldots, k_N$ and $l_0, l_1, \ldots l_N$. Doing so we find

$$E_{\mathcal{M}^\alpha}\Big(\alpha[\mathbf{i}; \mathbf{a}]\, \alpha[\mathbf{j}; \mathbf{b}]\Big) = E_{\mathcal{M}^\alpha}\Big(T_N(\alpha[\mathbf{i}; \mathbf{a}])\, T_N(\alpha[\mathbf{j}; \mathbf{b}])\Big)$$

for all $N > \max I \cup J$, where

$$T_N := \prod_{l=0}^{\rightarrow N} \alpha_l^{lN} M_l^{(N)} \quad \text{with } M_N^{(n)} := \frac{1}{n}\sum_{k=0}^{n-1} \alpha_N^k.$$

Since Corollary 8.7 ensures that all assumptions of the refined mean ergodic theorem Theorem 8.4 are satisfied, the pointwise SOT-convergence of T_N to $E_{\mathcal{M}^{\alpha_0}} (= E_{\mathcal{M}^\alpha})$ for $N \to \infty$ establishes

$$E_{\mathcal{M}^\alpha}\Big(\alpha[\mathbf{i};\mathbf{a}]\,\alpha[\mathbf{j};\mathbf{b}]\Big) = E_{\mathcal{M}^\alpha}\big(\alpha[\mathbf{i};\mathbf{a}]\big)\, E_{\mathcal{M}^\alpha}\big(\alpha[\mathbf{j};\mathbf{b}]\big)$$

for any $\mathbf{i}$ and $\mathbf{j}$ with disjoint ranges. This generalizes to the $\mathbb{C}$-linear span of monomials $\alpha[\mathbf{i}_n;\mathbf{a}_n]$ and $\alpha[\mathbf{j}_n;\mathbf{b}_n]$, provided the range of the tuples $\mathbf{i}_n$ is contained in I and the range of the tuples $\mathbf{j}_n$ is contained in J. Now a density argument establishes the factorization

$$E_{\mathcal{M}^\alpha}(xy) = E_{\mathcal{M}^\alpha}(x)E_{\mathcal{M}^\alpha}(y)$$

for all $x \in \mathcal{M}_I$ and $y \in \mathcal{M}_J$ whenever I and J are finite disjoint subsets of $\mathbb{N}_0$. Finally, another approximation removes the assumption of the finiteness of I and J. Thus we have established that the spreadability of a minimal stationary process $\mathscr{M}$ implies its full $\mathcal{M}^\alpha$-factorizability.

By Theorem 4.2, full $\mathcal{M}^\alpha$-factorizability and full $\mathcal{M}^\alpha$-independence are equivalent. In particular, we know already $\mathcal{M}^\alpha = \mathcal{M}^{\mathrm{tail}}$ from Theorem 7.2. Finally, Corollary 6.8 entails that $\mathscr{B}$ is a full Bernoulli shift. □

Remark 8.9. The refined version of the mean ergodic theorem, Theorem 8.4, is motivated in parts from product representations of endomorphisms as their study is started in [Goh04] and as they are applied to braid group representations in [GK08]. Suppose the probability space $(\mathcal{M}, \psi)$ is equipped with a tower

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$$

of ψ-expected subalgebras such that $\mathcal{M} = \bigvee_{n\geq 0} \mathcal{M}_n$ and consider a family of automorphisms $(\gamma_k)_{k\in\mathbb{N}} \subset \mathrm{Aut}(\mathcal{M},\psi)$ satisfying

$$\begin{aligned} \gamma_k(\mathcal{M}_n) &= \mathcal{M}_n && \text{if } k \leq n \\ \gamma_k|_{\mathcal{M}_{n-1}} &= \mathrm{id}\,|_{\mathcal{M}_{n-1}} && \text{if } k \geq n+1. \end{aligned}$$

Then

$$\alpha_N := \lim_{n\to\infty} \gamma_{N+1}\cdots\gamma_n$$

exists in the pointwise strong operator topology and defines a family of ψ-preserving endomorphisms $\{\alpha_N\}_{N\in\mathbb{N}_0}$ of $\mathcal{M}$ such that $\mathcal{M}_N \subset \mathcal{M}^{\alpha_N} \subset \mathcal{M}^{\alpha_{N+1}}$ for all $N \in \mathbb{N}_0$.

Suppose now in addition that

$$\alpha_N|_{\alpha_0^k(\mathcal{M})} = \alpha_0|_{\alpha_0^k(\mathcal{M})} \quad \text{if } k \geq N.$$

Then it can be seen that the refined mean ergodic theorem preserves localization properties with respect to the filtration $(\mathcal{A}_I)_{I\subset\mathbb{N}_0}$, where $\mathcal{A}_I := \bigvee_{i\in I}\alpha_0^i(\mathcal{M}_0)$. To be more precise, suppose $x \in \mathcal{A}_I$ and $y \in \mathcal{A}_J$ with $I \cap J = \emptyset$. Then for every N, there exist sets I_N, J_N with $I_N \cap J_N = \emptyset$ such that $T_N(x) \in \mathcal{A}_{I_N}$ and $T_N(y) \in \mathcal{A}_{J_N}$. Such a feature turned out to be crucial for the proof that spreadability implies conditional full independence.

9. Some applications and outlook

We briefly address some further developments and applications of Theorem 0.2.

9.1. **A glimpse on braidability.**

The Artin's braid group $\mathbb{B}_\infty$ is presented by the generators $\sigma_1, \sigma_2, \ldots$, subject to the relations

$$\begin{aligned} \sigma_i\sigma_j\sigma_i &= \sigma_j\sigma_i\sigma_j && \text{if } \mid i-j \mid = 1, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i && \text{if } \mid i-j \mid > 1. \end{aligned}$$

$\mathbb{B}_n$ is an important extension of the symmetric group $\mathbb{S}_n$ and we introduce in [GK08] 'braidability' as a notion which extends exchangeability.

Definition 9.1. A random sequence $\mathscr{I}$ with random variables

$$\iota \equiv (\iota_n)_{n\geq 0} \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{M}, \psi)$$

is *ρ-braidable* if there exists a representation $\rho\colon \mathbb{B}_\infty \to \mathrm{Aut}(\mathcal{M}, \psi)$ satisfying:

$$\begin{aligned} \iota_n &= \rho(\sigma_n\sigma_{n-1}\cdots\sigma_1)\iota_0 && \text{for all } n \geq 1; \\ \iota_0 &= \rho(\sigma_n)\iota_0 && \text{if } n \geq 2. \end{aligned}$$

Note that the representation ρ may be non-faithful and comprises representations of $\mathbb{S}_\infty$. More precisely, it is shown in [GK08] that the following are equivalent:

(i) $\mathscr{I}$ is exchangeable;
(ii) $\mathscr{I}$ is ρ-braidable and $\rho(\sigma_n^2) = \mathrm{id}$ for all $n \in \mathbb{N}$.

So exchangeability clearly implies braidability. A main result of [GK08] is that braidability is intermediate between two distributional symmetries and thus provides a refinement of the noncommutative extended de Finetti theorem, Theorem 0.2:

Theorem 9.2 ([GK08]). *Let $\mathscr{I}$ be an infinite random sequence and consider the following statements:*

(a) *$\mathscr{I}$ is exchangeable;*
(ab) *$\mathscr{I}$ is braidable;*
(b) *$\mathscr{I}$ is spreadable;*
(c) *$\mathscr{I}$ is stationary and full $\mathcal{M}^{\mathrm{tail}}$-independent.*

Then we have the implications:

$$(a) \Rightarrow (ab) \Rightarrow (b) \Rightarrow (c)$$

Starting from braid group representations, this result implies a rich structure of triangular arrays of commuting squares, similar as they emerge from the Jones fundamental construction in subfactor theory. We refer the interested reader to [GK08] for further details and developments.

We need another result from [GK08] to complete the proof of Theorem 0.2.

Theorem 9.3 ([GK08]). *There exist examples of infinite random sequences such that the implications (a) $\Leftarrow$ (b)' and '(b) $\Leftarrow$ (c)' fail in Theorem 0.2 resp. Theorem 9.2.*

Proof. See Theorem 5.6, Theorem 5.9, Example 6.1 and Example 6.4 in [GK08]. □

9.2. **The prototype of a noncommutative conditioned central limit law.** Another immediate application of Theorem 0.2 is given by noncommutative central limit theorems. They are an integral component of quantum probability [CH71, Hud73, GvW78, vW78, Qua84] and free probability [Voi85, Voi86, Spe90, Voi91, VDN92]. Unified general versions of them are obtained in the setting of *-algebraic probability spaces in [Spe92, SvW94] and the related algebraic techniques are of growing interest in operator algebras. Especially Speicher's interpolation technique for q-commutation relations [Spe92] is successfully applied for results on hypercontractivity in [Bia97, Kem05] and the embedding of Pisier's operator Hilbert space OH into the predual of the hyperfinite III_1 factor due to Junge [Jun06].

To control the existence of a limit distribution in a *-algebraic setting, general limit theorems need to stipulate three more or less technical conditions on mixed moments of the random variables: a singleton condition, a growth condition and some appropriate form of order-invariance condition on second order correlations [SvW94]. These three conditions have been replaced by two conditions in [KS07] when working with tracial W*-algebraic probability spaces: a growth condition and order-invariance (which equals 'spreadability' herein). This leads to precise formulas for the higher moments of additive flows with stationary independent increments whenever they are spreadable. An application of Theorem 0.2 allows us to show that additive flows with spreadable increments have automatically independent stationary increments. In particular, one obtains for such additive flows a noncommutative generalization of [Kal05, Theorem 1.15], the continuous version of the extended de Finetti theorem. Related results will be published elsewhere.

Let us present here only a simple version of the central limit theorem for spreadable random sequences, the 'discrete time' analogue of spreadable additive flows. We need to introduce some notation for its formulation.

Let $\mathcal{O}(p)$ denote the set of equivalence classes $[\mathbf{i}]$ for p-tuples $\mathbf{i}\colon \{1,2,\ldots,p\} \to \mathbb{N}_0$ under the following equivalence relation: two p-tuples $\mathbf{i}$ and $\mathbf{j}$ are order equivalent if

$$\mathbf{i}(k) \leq \mathbf{i}(l) \Leftrightarrow \mathbf{j}(k) \leq \mathbf{j}(l) \qquad \text{for all } k,l = 1,\ldots,p.$$

Furthermore, let

$$\mathcal{O}_2(p) := \{[\mathbf{i}] \in \mathcal{O}(p) \,|\, |\mathbf{i}^{-1}(k)| \in \{0,2\}, k \in \mathbb{N}_0\},$$

the set of all equivalence classes of p-tuples with pair partitions as pre-image and let $P_2(p)$ denote the set of all pair partitions of $\{1,2,\cdots,p\}$. Note that $P_2(p)$ has the cardinality $p\,!! = (p-1)(p-3)\cdots 5\cdot 3\cdot 1$ for p even and $p\,!! = 0$ for p odd and that $|\mathcal{O}_2(p)|$, the cardinality of $\mathcal{O}_2(p)$, satisfies

$$p\,!! = \frac{|\mathcal{O}_2(p)|}{(p/2)!}.$$

The following result can be easily deduced from [KS07, Theorem 4.4], since condition (d) of Theorem 0.2 implies the vanishing of so-called 'singletons'.

Theorem 9.4. *Let the spreadable random sequence $\mathscr{I}$ be given by the random variables $(\iota_n)_{n\geq 0}\colon (\mathcal{M}_0,\psi_0) \to (\mathcal{M},\psi)$ and consider*

$$S_N(x) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \iota_n(x)$$

for some fixed $x \in \mathcal{M}_0$ with $E_{\mathcal{M}^{\mathrm{tail}}}(x) = 0$. Then

$$\lim_{N\to\infty} \psi(S_N(x)^p) = p\,!! \cdot a_p(x)$$

with the average

$$a_p(x) := \begin{cases} \dfrac{1}{|\mathcal{O}_2(p)|} \displaystyle\sum_{[\mathbf{i}]\in\mathcal{O}_2(p)} \psi\big(\iota_{\mathbf{i}(1)}(x)\iota_{\mathbf{i}(2)}(x)\cdots\iota_{\mathbf{i}(p)}(x)\big) & \text{for even } p, \\ 0 & \text{for odd } p. \end{cases}$$

This result can be regarded as the prototype of a noncommutative version of conditional central limit theorems in classical probability. We refer the reader to [DM02] for more information on this matter. Note also that above theorem can be promoted to an operator equation:

$$\text{SOT-}\lim_{N\to\infty} E_{\mathcal{M}^{\mathrm{tail}}}(S_N(x)^p) = p\,!! \cdot A_p(x)$$

with the average

$$A_p(x) := \begin{cases} \dfrac{1}{|\mathcal{O}_2(p)|} \displaystyle\sum_{[\mathbf{i}]\in\mathcal{O}_2(p)} E_{\mathcal{M}^{\mathrm{tail}}}\big(\iota_{\mathbf{i}(1)}(x)\iota_{\mathbf{i}(2)}(x)\cdots\iota_{\mathbf{i}(p)}(x)\big) & \text{for even } p, \\ 0 & \text{for odd } p. \end{cases}$$

Let us discuss more in detail the example that the $\iota_k(x)$'s mutually commute for fixed x. Then the averages $a_{2p}(x)$ and $A_{2p}(x)$ can be easily computed by Theorem 0.2 and the module property of conditional expectations:

$$\begin{aligned} a_{2p}(x) &= \psi\big(E_{\mathcal{M}^{\mathrm{tail}}}(x^2)^p\big), \\ A_{2p}(x) &= E_{\mathcal{M}^{\mathrm{tail}}}(x^2)^p. \end{aligned}$$

If the tail algebra $\mathcal{M}_{\mathrm{tail}}$ is trivial, we obtain the normal distribution as central limit law, since then $a_{2p}(x) = \psi(x^2)^p = a_2(x)^p$ and thus

$$\lim_{N\to\infty} \psi(S_N(x)^{2p}) = (2p)!! \cdot a_2(x)^p.$$

But if $\mathcal{M}_{\mathrm{tail}}$ is non-trivial, the limit law is different from the normal distribution; it is a mixture of them.

There seems to be an interesting connection to interacting Fock space models (as introduced in [AB98, ACL05]) in the conditional case. Given $x^* = x \in \mathcal{M}_0$ with $E_{\mathcal{M}^{\mathrm{tail}}}(x) = 0$ and $E_{\mathcal{M}^{\mathrm{tail}}}(x^2) \neq 0$ in the setting of above example, there exists a monotone increasing sequence $(\lambda_{2p})_p$ with $\lambda_2 = 1$ such that, for all p,

$$a_{2p}(x) = \lambda_{2p} a_2(x)^p$$

Here the properties of $(\lambda_{2p})_p$ are deduced from the fact that $L^p(\mathcal{M}_{\mathrm{tail}}, \psi|_{\mathcal{M}^{\mathrm{tail}}})$ isomorphic to a classical L^p-space (w.r.t. some probability measure). Now $\lambda_{2p+2} \geq \lambda_{2p}$ is concluded from the monotony of the L^p-norms.

Already this simple class of examples hints at that non-trivial tail algebras lead to interesting examples of interacting Fock space models through central limit techniques, such that the limit object '$\lim_{N\to\infty} S_N(x)$' reappears as the sum of creation and annihilation operator on an appropriately chosen interacting Fock space.

Moreover, it is worthwhile to mention that the central limit law is Wigner's semicircle law if the averages $a_p(x)$ are connected to the second order moment

$\psi(x^2)$ by the formula

$$a_{2p}(x) = \frac{C_p}{(2p)!!}\psi(x^2)^p,$$

whenever $E_{\mathcal{M}_{\mathrm{tail}}}(x) = 0$ and $\psi(x^2) \neq 0$. Here C_p denotes the p-th Catalan number.

The amazing analogy of results in classical probability and free probability prompts of course the question if the condition

$$A_{2p}(x) = \frac{C_p}{(2p)!!}E_{\mathcal{M}^{\mathrm{tail}}}(x^2)^p$$

can be better understood in the context of freeness with amalgamation.

At this stage of our knowledge we regard it to be of major interest to identify concrete central limit laws which can emerge from spreadable random sequences. This line of research is continued in [GK08], where we will investigate central limit laws in the context of braid group representations as stated in Theorem 9.2. At the time of this writing we have strong numerical evidence that the spectral distributions of q-Gaussian random variables are among the central limit laws for random sequences constructed on simple examples of Jones towers on the hyperfinite II_1 factor.

9.3. **Noncommutative L^p-inequalities for spreadable random sequences.** As a third application we address Junge's L^1-inequality for systems of independent, conditioned top-subsymmetric copies of a von Neumann algebra [Jun06, Theorem 1.1]. Top-subsymmetry is a slight generalization of subsymmetry or, in our formulation, spreadability. By Theorem 0.2, the assertion of independence can be dropped in the context of spreadability.

Given the random sequence $\mathscr{I}$, we identify the probability space $(\mathcal{A}_0, \psi_0)$ with $(\mathcal{M}_0, \psi_0) := (\iota_0(\mathcal{A}_0), \psi|_{\iota_0(\mathcal{A}_0)})$ and thus have $\iota_0(x) = x$ for all $x \in \mathcal{M}_0$. The endomorphisms ι_k extend to isometric embeddings from $L^1(\mathcal{M}_0)$ into $L^1(\mathcal{M})$, the Haagerup L^1-spaces, and are denoted by the same symbol. Similarly, the state-preserving conditional expectation from $\mathcal{M}$ onto $\mathcal{M}_{\mathrm{tail}}$ extends to a projection from $L^1(\mathcal{M})$ onto $L^1(\mathcal{M}_{\mathrm{tail}})$, in the following just denoted by E. We refer the reader for further information on the technical details to [Jun06] and the references cited therein. The main inequality of [Jun06] can now be reformulated as follows. We are indebted to Junge who pointed out to the author this immediate reformulation.

Theorem 9.5. *Suppose $\mathscr{I}$ is a spreadable random sequence with above identification and let $x \in L^1(\mathcal{M}_0)$ with $E(x) = 0$. Then, for all $n \in \mathbb{N}$,*

$$\Big\| \sum_{k=0}^{n-1} \iota_k(x) \Big\|_1 \sim \inf_{x = x_1 + x_2 + x_3} n\|x_1\|_1 + \sqrt{n}\|E(x_2^* x_2)^{1/2}\|_1 + \sqrt{n}\|E(x_3 x_3^*)^{1/2}\|_1.$$

Here $a \sim b$ means that there exists an absolute constant $c > 0$ such that $c^{-1}a \leq b \leq ca$. This constant is independent of n and x in the above stated theorem. A corollary of this inequality is the following estimate:

$$\lim_{n\to\infty} \Big\| \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \iota_k(x) \Big\|_1 \sim \inf_{x = x_2 + x_3} \|E(x_2^* x_2)^{1/2}\|_1 + \|E(x_3 x_3^*)^{1/2}\|_1.$$

Of course, a further immediate application is given by noncommutative Rosenthal inequalities of Junge and Xu [JX03]. They established the noncommutative version of inequalities for the p-norm of independent mean-zero random variables found by Rosenthal [Ros70]. With Theorem 0.2 at our hands, spreadable random sequences produce a rich class of new examples. The noncommutative Rosenthal inequalities are even of interest for independent copies of a single random variable since we have still a very incomplete picture on the resulting central limit laws.

Theorem 9.6. *Let $2 \leq p < \infty$. Suppose $\mathscr{I}$ is a spreadable random sequence and let $(x_n)_{n\geq 0} \subset L^p(\mathcal{M}_0)$ with $E(x_n) = 0$ for all n. Then there exist universal constants δ_p and η_p such that,*

$$\delta_p^{-1}\, s_{p,n}(x) \leq \Big\|\sum_{k=0}^{n-1} \iota_k(x_k)\Big\|_p \leq \eta_p\, s_{p,n}(x),$$

where

$$s_{p,n}(x) = \max\left\{ \|\big(\sum_{k=0}^{n-1} |\iota_k(x_k)|^p\big)^{1/p}\|_p, \|\big(\sum_{k=0}^{n-1} E(x_k^* x_k)\big)^{1/2}\|_p, \|\big(\sum_{k=0}^{n-1} E(x_k x_k^*)\big)^{1/2}\|_p \right\}.$$

We note that a similar inequality is valid for $1 < p < 2$ (see [JX03, Theorem 6.1]). In the special case of constant selfadjoint sequences, i.e. $x_n = x$, the above inequality yields

$$\lim_{n\to\infty} \big\|\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \iota_k(x)\big\|_p \sim_p \max\left\{ \|E(x^*x)^{1/2}\|_p, \|\big(E(xx^*)^{1/2}\|_p \right\}.$$

Here $a \sim_p b$ means that there exists a constant c_p such that $c_p^{-1} a \leq b \leq c_p a$.

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