A dual de Finetti theorem

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The quantum de Finetti theorem says that, given a symmetric state, the state obtained by tracing out some of its subsystems approximates a convex sum of power states. The more subsystems are traced out, the better this approximation becomes. Schur-Weyl duality suggests that there ought to be a dual result that applies to a unitarily invariant state rather than a symmetric state. Instead of tracing out a number of subsystems, one traces out part of every subsystem. The theorem then asserts that the resulting state approximates the fully mixed state, and the larger the dimension of the traced-out part of each subsystem, the better this approximation becomes. This paper gives a number of propositions together with their dual versions, to show how far the duality holds.

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I. INTRODUCTION

Suppose we have a state space $H = (\mathbb{C}^d)^{\otimes n}$ consisting of n identical subsystems. The quantum de Finetti theorem [1, 2] tells us that, given a symmetric state on H, the state obtained by tracing out $n-k$ of the subsystems can be approximated by a convex sum of powers, i.e. by a convex sum of states of the form $\sigma^{\otimes k}$; the smaller k/n, the better the approximation. This is a useful result, because such power states are often rather easy to analyse.

Now, the symmetric group S_n and the unitary group $\mathcal{U}(d)$ both act on the space $(\mathbb{C}^d)^{\otimes n}$, the former by permuting the factors and the latter by applying any $g \in \mathcal{U}(d)$ to each factor, so the action is given by $g^{\otimes n}$. These actions commute, and this leads to a type of duality, called Schur-Weyl duality [3]. Given any result that holds for the symmetric group, one can hope to find a dual result for the unitary group.

Here I show that there is a dual to the de Finetti theorem, obtained by swapping the roles of S_n and $\mathcal{U}(d)$. The situation is summed up in Table I. Instead of symmetric states, we consider unitarily-invariant states. And instead of tracing out a number of subsystems, we trace out part of each subsystem; more precisely, we replace each individual subsystem $\mathbb{C}^d$ by $\mathbb{C}^p \otimes \mathbb{C}^q$, and we trace out the $\mathbb{C}^q$ part from all the subsystems in $(\mathbb{C}^p \otimes \mathbb{C}^q)^{\otimes n}$. The theorem then states that, when q is large relative to n, the resulting traced-out state approximates the fully mixed state. This is different in character from the standard de Finetti theorem, in that all information about the original state is lost. However, this fact in itself may lead to some interesting applications.

As far as possible, the results are laid out as pairs of propositions that are duals of each other. Some of these pairs are exact analogues; in other cases, one of the pair is less meaningful or even trivial. This gives some insight into the nature of the duality.

TABLE I:

Standard de Finetti theorem.	**Dual theorem.**
Symmetric state ρ.	Unitarily-invariant state ρ.
State space is $(\mathbb{C}^d)^{\otimes n}$.	State space is $(\mathbb{C}^p \otimes \mathbb{C}^q)^{\otimes n}$.
Trace out $n-k$ subsystems.	Trace out $\mathbb{C}^q$ from each subsystem.
$\mathrm{tr}_{n-k}\rho \approx$ convex sum of powers.	$\mathrm{tr}_{\mathbb{C}^q}\rho \approx$ fully mixed state.

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II. DUALITY FOR SYMMETRIC WERNER STATES.

We will refer to unitarily-invariant states as *Werner* states [4]. Rather than considering general Werner states, we begin by looking at a special class, the symmetric Werner states, i.e. states that are invariant under both the unitary and symmetric groups. The de Finetti theorem and its dual can then be applied to the same state, so the pattern becomes particular clear, as shown in Table II.

The Schur-Weyl decomposition [5] of $H = (\mathbb{C}^d)^{\otimes n}$ is given by:

$$(\mathbb{C}^d)^{\otimes n} \cong \bigoplus_{\lambda \in Par(n,d)} U_\lambda \otimes V_\lambda, \tag{1}$$

where U_λ is the irrep (irreducible representation) of $\mathcal{U}(d)$ with highest weight $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$, and V_λ is the irrep of S_n defined by the same partition λ. Here $Par(n,d)$ denotes the ordered partitions of n with at most d rows. We will also refer to a $\lambda \in Par(n,d)$ as a (Young) diagram.

Let P_λ denote the projector onto the subspace $U_\lambda \otimes V_\lambda$ in the Schur-Weyl decomposition. Write $f_\lambda = \dim(V_\lambda)$, and $e^d_\lambda = \dim(U_\lambda)$, where d is the dimension of the unitary group $\mathcal{U}(d)$. Then the normalised projector $\rho_\lambda = P_\lambda/(e^d_\lambda f_\lambda)$ is a symmetric Werner state, and in fact any symmetric Werner state ρ can be written as a weighted sum of such projectors, $\sum_\mu a_\mu \rho_\mu$, with $\sum_\mu a_\mu = 1$ [6]. Let $\mathsf{tr}_{n-k}\rho_\lambda$ denote the state obtained by tracing out $n-k$ of the n subsystems from the state ρ_λ. Lemma III.4 in [6] can be restated as follows:

Proposition II.1 (Trace formula)**.** *Let $\lambda \in Par(n,d)$. Then*

$$\mathsf{tr}_{n-k}\rho_\lambda = \frac{1}{f_\lambda} \sum_\mu \rho_\mu f_\mu \left(\sum_\nu c^\lambda_{\mu\nu} f_\nu \right),$$

where the sums extends over all $\mu \in Par(k,d)$ and $\nu \in Par(n-k,d)$, and $c^\lambda_{\mu\nu}$ is the Littlewood-Richardson coefficient, i.e. the coefficient in the Clebsch-Gordan series for $\mathcal{U}(d)$: $U_\mu \otimes U_\nu = \sum_\lambda c^\lambda_{\mu\nu} U_\lambda$.

From now on, we assume each individual subsystem $\mathbb{C}^d$ is bipartite, so it can be written as $\mathbb{C}^p \otimes \mathbb{C}^q$. Let $\mathsf{tr}_{\mathbb{C}^q}$ denote the result of tracing out $\mathbb{C}^q$ from each subsystem in the total state space $(\mathbb{C}^p \otimes \mathbb{C}^q)^{\otimes n}$. The dual of the preceding Proposition is:

Proposition II.2 (Dual trace formula)**.** *Let $\lambda \in Par(n,pq)$. Then*

$$\mathsf{tr}_{\mathbb{C}^q}\rho_\lambda = \frac{1}{e^{pq}_\lambda} \sum_\mu \rho_\mu e^p_\mu \left(\sum_\nu g_{\lambda\mu\nu} e^q_\nu \right),$$

where the sums extend over all diagrams $\mu \in Par(n,p)$ and $\nu \in Par(n,q)$, and $g_{\lambda\mu\nu}$ is the Kronecker coefficient, i.e. the coefficient in the Clebsch-Gordan series for S_n: $V_\mu \otimes V_\nu = \sum_\lambda g_{\lambda\mu\nu} V_\lambda$.

Proof. We can restrict the action of the group $\mathcal{U}(pq)$ on $\mathbb{C}^p \otimes \mathbb{C}^q$ to the subgroup $\mathcal{U}(p) \times \mathcal{U}(q)$. This gives an expansion in tensor products of irreps [7]:

$$U_\lambda = \sum_{\mu\nu} g_{\lambda\mu\nu} U_\mu \otimes U_\nu,$$

where $\mu \in Par(n,p)$ and $\nu \in Par(n,q)$. If P_{U_λ} denotes the projector onto U_λ, we can rewrite this as

$$P_{U_\lambda} = \sum_{\mu\nu} \sum_{i=1}^{g_{\lambda\mu\nu}} P^i_{U_\mu} \otimes P^i_{U_\nu}. \tag{2}$$

Taking the trace over $\mathbb{C}^q$ gives

$$\mathsf{tr}_{\mathbb{C}^q} P_{U_\lambda} = \sum_{\mu\nu} \sum_{i=1}^{g_{\lambda\mu\nu}} P^i_{U_\mu} e^q_\nu. \tag{3}$$

Now define the symmetric average, $\mathbb{S}$, by

$$\mathbb{S}(\tau) = \frac{1}{n!} \sum_{\pi \in S_n} \pi\tau\pi^{-1}, \tag{4}$$

TABLE II:

Duality Dictionary for symmetric Werner states.	
f_λ $(\dim V_\lambda)$.	e^d_λ $(\dim U_\lambda)$.
Littlewood-Richardson coefficient $c^\lambda_{\mu\nu}$.	Kronecker coefficient $g_{\lambda\mu\nu}$.
Unitary group character (Schur function) s_λ.	Symmetric group character χ^λ.
Shifted Schur function $s^*_\mu(\lambda)$.	Character polynomial $\chi^{\lambda\mu}(q)$ (Definition II.4).
Twirled power state.	Symmetrised cycle operator.

for any operator τ. Applying $\mathbb{S}$ to both sides of (3), Schur's lemma implies

$$\mathrm{tr}_{\mathbb{C}^q}\frac{P_\lambda}{f_\lambda} = \sum_\mu \frac{P_\mu}{f_\mu}\left(\sum_\nu g_{\lambda\mu\nu}e^q_\nu\right).$$

Substituting $\rho_\lambda = P_\lambda/(e^{pq}_\lambda f_\lambda)$, $\rho_\mu = P_\mu/(e^p_\mu f_\mu)$ gives the result we seek. □

This shows, incidentally, why the dual operation to tracing out over $n-k$ subsystems is to trace out over part of each subsystem: the analogue of the subgroup $S_k \times S_{n-k} \subset S_n$ is the subgroup $\mathcal{U}(p)\times\mathcal{U}(q) \subset \mathcal{U}(pq)$.

Theorem 8.1 in [8] allows one to evaluate the bracketted inner sum in Proposition II.1. We restate this result as follows:

Proposition II.3 (Inner sum formula)**.**

$$\sum_\nu c^\lambda_{\mu\nu} f_\nu = \frac{f_\lambda s^*_\mu(\lambda)}{n\downharpoonright k},$$

where $s^*_\mu(\lambda)$ is the shifted Schur function defined in [8] and $n \downharpoonright k = n(n-1)\dots(n-k+1)$.

Likewise, one can evaluate the bracketed inner sum in Proposition II.2. First we introduce a symmetric-group analogue of the shifted Schur function:

Definition II.4. *Suppose λ and μ are arbitrary diagrams with n boxes. The* character polynomial *$\chi^{\lambda\mu}(q)$ is the polynomial in q defined by*

$$\chi^{\lambda\mu}(q) = \sum_{\pi\in S_n} q^{c(\pi)}\chi^\lambda(\pi)\chi^\mu(\pi),$$

where $\chi^\mu(\pi)$ is the character of the symmetric group evaluated at the permutation π and $c(\pi)$ is the number of cycles in π.

The character polynomial can sometimes be more conveniently calculated by summing over cycle types α rather than permutations, giving

$$\chi^{\lambda\mu}(q) = \sum_{\alpha\in Par(n,n)} h_\alpha q^{c(\alpha)}\chi^\lambda(\alpha)\chi^\mu(\alpha),$$

where h_α is the number of elements in the conjugacy class α [9], and $c(\alpha)$ is the number of rows in the diagram α representing the cycle type.

Proposition II.5 (Dual inner sum formula)**.**

$$\sum_\nu g_{\lambda\mu\nu}e^q_\nu = \frac{\chi^{\lambda\mu}(q)}{n!}.$$

Proof. First observe that

$$e^q_\nu = \frac{1}{n!}\sum_{\pi\in S_n} q^{c(\pi)}\chi^\nu(\pi). \tag{5}$$

This follows from the fact [9] that the projector P_ν on $(\mathbb{C}^q)^{\otimes n}$ is defined by

$$P_\nu = \frac{f_\nu}{n!}\sum_\pi \chi^\nu(\pi)\pi,$$

and it vanishes on all components of the Schur-Weyl decomposition (1) except $U_\nu \otimes V_\nu$, where it has trace $e^q_\nu f_\nu$. On the other hand, the trace of π acting on $(\mathbb{C}^q)^{\otimes n}$ is given by $q^{c(\pi)}$ since the basis elements $e_{i_1}\otimes \ldots \otimes e_{i_n}$ of $(\mathbb{C}^q)^{\otimes n}$ that are fixed by π, i.e. that contribute to $\mathsf{tr}\pi$, are those that assign the same e_i to all the elements of each cycle of π, and there are q ways of picking an e_i and $c(\pi)$ cycles. Thus P_ν has trace $\frac{f_\nu}{n!}\sum_\pi q^{c(\pi)}\chi^\nu(\pi)$, and equating these two expressions for the trace gives (5).

Now the Kronecker coefficient can be defined by

$$g_{\lambda\mu\nu} = \frac{1}{n!}\sum_\pi \chi^\lambda(\pi)\chi^\mu(\pi)\chi^\nu(\pi).$$

Combining this with (5), we have

$$\sum_\nu g_{\lambda\mu\nu}e^q_\nu = \frac{1}{n!}\sum_{\pi,\pi'} q^{c(\pi')}\chi^\lambda(\pi)\chi^\mu(\pi)\left(\frac{1}{n!}\sum_\nu \chi^\nu(\pi)\chi^\nu(\pi')\right).$$

The orthogonality relations for characters imply that the expression in brackets is zero if π and π' are in different conjugacy classes, and is otherwise the inverse of $h_{[\pi]}$, the number of elements in the conjugacy class of π. As $c(\pi')$ only depends on the conjugacy class of π', the result follows. □

Propositions II.3 and II.1 can be used to prove the de Finetti theorem for symmetric Werner states [6]:

Theorem II.6 (de Finetti theorem). *Let ρ_λ be the normalised projector onto the Young subspace of $(\mathbb{C}^d)^{\otimes n}$ with diagram λ. Then*

$$||\mathsf{tr}_{n-k}\rho_\lambda - \tau|| \le \frac{3}{4}\cdot\frac{k(k-1)}{\lambda_\ell} + O(\frac{k^4}{\lambda_\ell^2}) \ ,$$

where τ is a convex sum of power states and λ_ℓ is the smallest non-zero component of λ.

Theorem II.7 (Dual de Finetti theorem).

$$||\mathsf{tr}_{C^q}\rho_\lambda - \frac{\mathcal{I}}{p^n}|| \le 2 - 2\left(\frac{q-n+1}{q}\right)^n = \frac{2n(n-1)}{q} + O(n^4/q^2),$$

where $\mathcal{I}$ is the identity on $(\mathbb{C}^p)^{\otimes n}$.

We leave the proof till section V, where the theorem is proved for all Werner states, not just symmetric ones.

Example II.8. *The simplest example is the symmetric subspace for $n = 2$. Using Propostions II.2 and II.5, we find*

$$\mathsf{tr}_{\mathbb{C}^q}\rho_{(2)} = \frac{(p+1)(q+1)}{2(pq+1)}\rho_{(2)} + \frac{(p-1)(q-1)}{2(pq+1)}\rho_{(1^2)}.$$

Also

$$\frac{\mathcal{I}}{p^2} = \frac{(p+1)}{2p}\rho_{(2)} + \frac{(p-1)}{2p}\rho_{(1^2)},$$

from which one gets

$$||\mathsf{tr}_{\mathbb{C}^q}\rho_{(2)} - \frac{\mathcal{I}}{p^2}|| = \frac{p^2-1}{p^2q+p}.$$

Note that the bound tends to zero with $q \to \infty$ but its behaviour does not depend sensitively upon p; in particular, there is no requirement for p to be small relative to q (see Discussion).

III. TWIRLED POWER STATES AND THEIR DUALS

Theorem II.6 in [6] actually makes the stronger claim that the approximating state τ is the twirl of a power state σ^k. We describe this now and also its dual version, where the analogue of the power state is a permutation matrix. However, the rewards of the dual approach diminish rapidly, and one does not get a stronger version of Theorem II.7 as will become clear at the end of this section.

Let us define the twirl of an arbitrary state τ on $(\mathbb{C}^d)^{\otimes k}$ as follows:

$$\mathbb{T}(\tau) = \int U^{\otimes k}\tau(U^\dagger)^{\otimes k} dU,$$

where dU is the Harr measure. Suppose $r = (r_1, \dots, r_d)$ is the spectrum of a state σ on $\mathbb{C}^d$. Then the *twirled power state* $\tau(r) = \mathbb{T}(\sigma^{\otimes k})$ depends only on r and not on the particular state σ chosen. Lemma III.1 from [6] expresses $\tau(r)$ in terms of basic Werner states.

Proposition III.1 (Twirl sum)**.** *Given a spectrum* $r = (r_1, \dots, r_d)$*,*

$$\tau(r) = \sum_\mu f_\mu s_\mu(r)\rho_\mu,$$

where $s_\mu(r)$ *is the Schur function.*

To define the dual version of a twirled power state, let π be a permutation, and let $b_1, \dots b_d$ be a basis in $\mathbb{C}^d$. Define the permutation matrix τ_π by $\tau_\pi = \pi\mathcal{I}$, i.e.

$$\tau_\pi = \sum_{0\le i_1 \dots i_n \le d} |(b_{i_{\pi(1)}}, \dots, b_{i_{\pi(n)}})\rangle\langle(b_{i_1}, \dots, b_{i_n})|.$$

Let λ be a Young diagram with n boxes representing a permutation cycle type. Pick any permutation π with cycle type λ. The *symmetrised cycle operator* $\sigma(\lambda)$ is defined to be $\mathbb{S}(\tau_\pi/d^n)$, where $\mathbb{S}$ is defined by (4). This does not depend on the choice of a permutation π having the cycle type λ. We can regard $\mathbb{S}$ as the "dual twirl", with the symmetric group replacing the unitary group. Thus we have:

Proposition III.2 (Dual twirl sum)**.** *Given a cycle type* λ*,*

$$\sigma(\lambda) = \frac{1}{d^n}\sum_\mu e^d_\mu \chi^\mu(\lambda)\rho_\mu. \tag{6}$$

Proof. By construction, $\sigma(\lambda)$ is symmetric; it is also unitarily invariant, since $U\tau_\pi U^\dagger = U\pi I U^\dagger = \pi U I U^\dagger = \pi I = \tau_\pi$. Thus $\sigma(\lambda)$ can be expressed as a sum $\sum_\mu c_\mu \rho_\mu$, where the coefficients are given by

$$c_\mu = \mathsf{tr}[P_\mu \sigma(\lambda)] = \mathsf{tr}[P_\mu \mathbb{S}(\tau_\pi/d^n)] = \mathsf{tr}[\mathbb{S}(P_\mu)\tau_\pi]/p^n = \mathsf{tr}[P_\mu\tau_\pi]/p^n,$$

π being a permutation of cycle type λ. But $\mathsf{tr}[P_\mu\tau_\pi]$ is the character of the representation $\pi \to P_\mu\tau_\pi P_\mu$, and as this is equivalent to e^d_μ copies of the irrep V_λ, we have $c_\lambda = e^d_\mu \chi^\mu(\lambda)/p^n$. $\square$

Note that $\sigma(\lambda)$ is in general not a state, since its eigenvalues, the coefficients in (6), can be negative. For instance, with $d = 3$, $\sigma((2,1)) = \frac{10}{27}\rho_{(3)} - \frac{8}{27}\rho_{(1^3)}$.

Returning to the standard de Finetti theorem, Propositions II.1 and II.3 tell us that, for $\lambda \in Par(n,d)$,

$$\mathsf{tr}_{n-k}\rho_\lambda = \sum_\mu \rho_\mu f_\mu \frac{s^*_\mu(\lambda)}{n \downharpoonright k},$$

The shifted Schur function [8], $s^*_\mu(\lambda)$, which appears on the right-hand side of this equation, is a polynomial in the λ_i, and its highest degree terms are the ordinary Schur function $s_\mu(\lambda)$. It follows that

$$\frac{s^*_\mu(\lambda)}{n \downharpoonright k} \to s_\mu(\bar\lambda) \text{ as } n \to \infty, \tag{7}$$

where $\bar\lambda = (\lambda_1/\sum\lambda_i, \dots, \lambda_d/\sum\lambda_i)$. Putting this together with Proposition III.1, we can restate Theorem II.6, showing that the approximating state can be taken to be the twirled power state $\tau(\bar\lambda)$.

Proposition III.3 (Twirl limit for de Finetti theorem)**.**

$$||\mathsf{tr}_{n-k}\rho_\lambda - \tau(\bar{\lambda})|| \le \frac{3}{4}\cdot\frac{k(k-1)}{\lambda_\ell} + O(\frac{k^4}{\lambda_\ell^2}) \ . \tag{8}$$

Dually, Propositions II.2 and II.5 tell us that

$$\mathsf{tr}_{\mathbb{C}^q}\rho_\lambda = \frac{1}{e_\lambda^{pq}}\sum_\mu \rho_\mu e_\mu^p \frac{\chi^{\lambda\mu}(q)}{n!}, \tag{9}$$

We can imitate the approximation of the shifted Schur function by the ordinary Schur function, and take the highest degree term in $\chi^{\lambda\mu}(q)$, which is $q^n\chi^\lambda(1^n)\chi^\mu(1^n)$. Using equation 6 and the fact that $\chi^\lambda(1^n) = f_\lambda$, we get

$$\mathsf{tr}_{\mathbb{C}^q}\rho_\lambda \to \frac{(pq)^n f_\lambda}{e_\lambda^{pq} n!}\sigma(1^n) \text{ as } q\to\infty.$$

We shall see later that the rather complicated coefficient of $\sigma(1^n)$ tends to 1 for large q (see inequality 13). This enables us to write

Proposition III.4 (Twirl limit for dual de Finetti theorem)**.**

$$||\mathsf{tr}_{C^q}\rho_\lambda - \sigma(1^n)|| \le \frac{2n(n-1)}{q} + O(n^4/q^2). \tag{10}$$

Unlike Proposition III.3, however, this adds nothing to the preceding result (Theorem II.7), since $\sigma(1^n) = \mathcal{I}/p^n$. A more interesting result is obtained from equation (9) without taking the limit of large q:

$$\mathsf{tr}_{\mathbb{C}^q}\rho_\lambda = \frac{1}{n! e_\lambda^{pq}}\sum_\pi q^{c(\pi)}\chi^\lambda(\pi)\sigma(\pi), \tag{11}$$

This shows how symmetrised cycle operators other than $\sigma(1^n)$ contribute to the trace.

IV. THE QUANTUM MARGINAL PROBLEM AND HORN'S CONJECTURE

We have now compared most of the ingredients of the de Finetti theorem and their dual versions. In this section we complete this process by comparing the shifted Schur functions that appear in Proposition II.3 with the character polynomials that appear in Proposition II.5. We do this by relating each of them to a mathematical problem of some historical interest. For the shifted Schur functions this is Horn's conjecture [10], whereas for the character polynomials it is the quantum marginal problem [11]. We begin with the latter.

Let $\rho_A = \mathsf{tr}_B(\rho_{AB})$ and $\rho_B = \mathsf{tr}_A(\rho_{AB})$ be the two marginal states of a bipartite state ρ_{AB}. Let $\Sigma^{p,q}$ denote the set of triples of spectra $\{\mathsf{Spec}(\rho_{\mathsf{AB}}), \mathsf{Spec}(\rho_{\mathsf{A}}), \mathsf{Spec}(\rho_{\mathsf{B}})\}$ for all operators ρ_{AB} on $\mathbb{C}^p\otimes\mathbb{C}^q$. It was shown in [7], [11], [12] that $\Sigma^{p,q}$ can be defined in terms of the Kronecker coefficients. Given a diagram λ, define $\bar{\lambda} = (\lambda_1/\sum\lambda_i, \dots \lambda_d/\sum\lambda_i)$, and let K be the set of all triples $(\bar{\lambda},\bar{\mu},\bar{\nu})$ with $\lambda\in Par(n,pq)$, $\mu\in Par(n,p)$, $\nu\in Par(n,q)$, for some n, satisfying $g_{\lambda\mu\nu} > 0$. Then $\Sigma^{p,q}$ is $\bar{K}$, the closure of K.

One can also focus on a single marginal, and ask which *pairs*, $\{\mathsf{Spec}(\rho_{\mathsf{AB}}), \mathsf{Spec}(\rho_{\mathsf{A}})\}$ of spectra can occur [13]. From the characterisation of $\Sigma^{p,q}$, it follows that this set, $\Gamma^{p,q}$ say, is the closure of the set of pairs $(\bar{\lambda},\bar{\mu})$ where $\lambda\in Par(n,pq)$, $\mu\in Par(n,p)$, and there is some $\nu\in Par(n,q)$ satisfying $g_{\lambda\mu\nu} > 0$. For a given λ, the μ's satisfying this condition correspond to the ρ_μ's that have non-zero coefficients in the expansion of $\mathsf{tr}_{\mathbb{C}^q}\rho_\lambda$ given by Proposition II.2. This, together with Proposition II.5, implies

Proposition IV.1 (Character polynomial condition for the marginal problem)**.** *Suppose $\lambda\in Par(n,pq)$, $\mu\in Par(n,p)$, and $\chi^{\lambda\mu}(q) > 0$. Then $(\bar{\lambda},\bar{\mu})\in\Gamma^{p,q}$.*

The converse does not follow from the characterisation of $\Sigma^{p,q}$ by Kronecker coefficients. If $\lambda\in Par(n,pq)$ and $\mu\in Par(n,p)$, and $(\bar{\lambda},\bar{\mu})\in\Gamma^{p,q}$ then we know there is a state ρ_{AB} with $\mathsf{Spec}(\rho_{\mathsf{AB}}) = \bar{\lambda}$ and $\mathsf{Spec}(\rho_{\mathsf{A}}) = \bar{\mu}$, but it does not follow that $\mathsf{Spec}(\rho_{\mathsf{B}})$ has the form $\bar{\nu}$ for some $\nu\in Par(n,q)$, or even that $\mathsf{Spec}(\rho_{\mathsf{B}})$ is rational. Even if it were true that $\mathsf{Spec}(\rho_{\mathsf{B}}) = \bar{\nu}$ with $\nu\in Par(n,q)$, we could only conclude [11, 12] that $g_{m\lambda\ m\mu\ m\nu} > 0$ for some integer $m > 0$ and hence that $\chi^{m\lambda\ m\mu}(q) > 0$ for some $m > 0$.

Proposition IV.2. *For any $\lambda \in Par(n, pq)$, $\mu \in Par(n, p)$, there is an integer q_+ in the range $1 \leq q_+ \leq n$ such that $\chi^{\lambda\mu}(q) > 0$ for $q \geq q_+$ and $\chi^{\lambda\mu}(q) = 0$ for $0 \leq q < q_+$. If $\lambda \neq \mu$, $q_+ \geq 2$.*

Proof. Clearly $\chi^{\lambda\mu}(q) = 0$ for $q = 0$, and as $\chi^{\lambda\mu}(q)$ is a polynomial of degree n and can therefore have at most n distinct roots, there must be some integer q in the range $1 \leq q \leq n$ for which $\chi^{\lambda\mu}(q) = 0$. Let q_+ be the least such q. Then by Proposition II.5, $\sum_\nu g_{\lambda\mu\nu} e_\nu^{q_+} > 0$, and thus $g_{\lambda\mu\nu} > 0$ and $e_\nu^{q_+} > 0$ for some ν. Thus $e_\nu^q > 0$ for all $q \geq q_+$, and $\chi^{\lambda\mu}(q) = \sum_\nu g_{\lambda\mu\nu} e_\nu^q > 0$ for all $q \geq q_+$. If $\lambda \neq \mu$, $\chi^{\lambda\mu}(1) = 0$ by the orthogonality relations for characters, so $q_+ \geq 2$. ◻

This result is also a consequence of a theorem of Berele and Imbo [14], which says that $g_{\lambda\mu\nu} > 0$ for some ν with $c(\nu) \leq \max\{c(\lambda), c(\mu)\}$. This implies the stronger result that $q_+ \leq \max\{c(\lambda), c(\mu)\}$.

Corollary IV.3. *For any $\lambda \in Par(n, pq)$, $\mu \in Par(n, p)$, there is an integer q_+ in the range $1 \leq q_+ \leq n$ such that $(\bar{\lambda}, \bar{\mu}) \in \Gamma^{p,q}$ for $q \geq q_+$.*

Example IV.4. *Take $\lambda = \mu$. Since every term in $\chi^{\lambda\lambda}(1)$ is non-negative, and the term with $\alpha = (1^n)$ is $f_\lambda^2/n! > 0$, we have $\chi^{\lambda\lambda}(1) > 0$ and hence $(\bar{\lambda}, \bar{\lambda}) \in \Gamma^{p,1}$. It is easy to see why this is true: take $\rho_{AB} = \rho_A \otimes |0\rangle\langle 0|_B$, and* $\mathsf{Spec}(\rho_{\mathsf{AB}}) = \mathsf{Spec}(\rho_{\mathsf{A}})$.

Example IV.5. *Take $\lambda = (1^n)$, $\mu = (n)$. Then $\chi^\lambda(\pi) = (-1)^{n+c(\pi)}$, by the Murnaghan-Nakayama rule [5], and $\chi^\mu(\pi) = 1$ for all π. It follows that*

$$\chi^{\lambda\mu}(q) = (-1)^n \sum_\pi (-q)^{c(\pi)} = q(q-1)\dots(q-n+1).$$

Thus $\chi^{\lambda\mu}(q) = 0$ for $q = 1, \dots, n-1$. Hence $(\bar{\lambda}, \bar{\mu}) \in \Gamma^{p,n}$. For $q \geq n$, a state with the appropriate spectra for ρ_{AB} and ρ_A is $\rho_{AB} = \frac{1}{n}|0\rangle\langle 0|_A \otimes \sum_{i=1}^n |i\rangle\langle i|_B$.

Since $\chi^{\lambda\mu}(q) = \chi^{\mu\lambda}(q)$, if $\lambda = (n)$, $\mu = (1^n)$ then $(\bar{\lambda}, \bar{\mu}) \in \Gamma_n$. A state with the appropriate spectra is $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$, where $\psi_{AB} = \frac{1}{\sqrt{n}}|11 + \dots + nn\rangle_{AB}$. (Note that this form of μ implies $p \geq n$.)

We can extend Proposition IV.2 as follows

Proposition IV.6. *For any $\lambda \in Par(n, pq)$, $\mu \in Par(n, p)$, there is a positive integer q_+ and a negative integer q_- such that $\chi^{\lambda\mu}(q) \neq 0$ for $q \geq q_+$ and $q \leq q_-$, and $\chi^{\lambda\mu}(q) = 0$ for $q_- < q < q_+$.*

Proof. Let λ' denote the diagram conjugate to λ, obtained by interchanging rows and columns. Then $\chi^{\lambda'}(\pi) = (-)^{n+c(\pi)}\chi^\lambda(\pi)$, so $\chi^{\lambda'\mu}(q) = (-1)^n \chi^{\lambda\mu}(-q)$. It follows that the negative range of integral roots has the same properties as the positive range, and the result follows from Proposition IV.2. ◻

Example IV.7. *Table III gives some examples of $\chi^{\lambda\mu}(q)$ for $n = 5$, illustrating the fact that the integral roots form a sequence without a gap. Note that $\chi^{\lambda'\mu'}(q) = \chi^{\lambda\mu}(q)$, since $\chi^{\lambda'}(\pi) = (-)^{n+c(\pi)}\chi^\lambda(\pi)$. To illustrate the property $\chi^{\lambda'\mu}(q) = (-1)^n\chi^{\lambda\mu}(-q)$, for each (λ, μ), either (λ', μ) or (λ, μ') is also given.*

For each λ, μ, states with $q = q_+$ and the appropriate spectra are described in Example IV.4 for the cases where $\lambda = \mu$, and in Example IV.5 for the case $(5), (1^5)$. States for the other cases are easy to construct; eg for $(4,1), (2,1^3)$, where $q_+ = 3$, we can take $p = 4$ and

$$\begin{aligned} \rho_{AB} &= \frac{1}{5}|11\rangle\langle 11|_{AB} + \frac{4}{5}|\psi_{AB}\rangle\langle\psi_{AB}|, \\ \text{where } |\psi\rangle_{AB} &= \frac{1}{2}|22 + 33\rangle_{AB} + \frac{1}{\sqrt{2}}|41\rangle_{AB}. \end{aligned}$$

Turning now to shifted Schur functions, Horn's conjecture – now a theorem [15] – states that, given $\lambda, \mu, \nu \in Par(n, d)$, $c_{\mu\nu}^\lambda > 0$ if and only if there is triple of $n \times n$ Hermitian matrices A, B and C with eigenvalues λ, μ and ν, respectively, such that $A + B = C$. Thus, if we know that $\sum_\nu c_{\mu\nu}^\lambda f_\nu > 0$, we can infer that

Proposition IV.8 (Shifted Schur function condition for Horn's conjecture). *Suppose $\lambda \in Par(n, d)$, $\mu \in Par(k, d)$. Then $s_\mu^*(\lambda) > 0$ implies that there are $n \times n$ Hermitian matrices A, B and C such that $A + B = C$ and the eigenvalues of C are λ_i, and those of A are μ_i.*

Proof. By Proposition II.3, $s_\mu^*(\lambda) > 0$ implies there is some $\nu \in Par(n-k, d)$ such that $c_{\mu\nu}^\lambda > 0$, and by the Horn-Klyachko theorem there are Hermitian matrices A, B, C with eigenvalues λ, μ, ν, respectively, satisfying $A + B = C$. ◻

TABLE III: Some examples of the polynomials $\chi^{\lambda\mu}(q)$ for $n = 5$.

λ, μ	$\chi^{\lambda\mu}(q)$	integral roots
$(5),(5)\ ;\ (1^5),(1^5)$	$q^5 + 10q^4 + 35q^3 + 50q^2 + 24q$	$-4,-3,-2,-1,0$
$(5),(4,1)\ ;\ (1^5),(2,1^3)$	$4q^5 + 20q^4 + 20q^3 - 20q^2 - 24q$	$-3,-2,-1,0,1$
$(4,1),(4,1)\ ;(2,1^3),(2,1^3)$	$16q^5 + 40q^4 + 20q^3 + 20q^2 + 24q$	$-2,-1,0$
$(4,1),(2,1^3)$	$16q^5 - 40q^4 + 20q^3 - 20q^2 + 24q$	$0,1,2$
$(5),(2,1^3);\ (1^5),(4,1)$	$4q^5 - 20q^4 + 20q^3 + 20q^2 - 24q$	$-1,0,1,2,3$
$(5),(1^5)$	$q^5 - 10q^4 + 35q^3 - 50q^2 + 24q$	$0,1,2,3,4$

Unlike Proposition IV.1, there is a simple criterion for the conditions of Proposition IV.8 to hold, since $s^*_\mu(\lambda) > 0$ if and only if $\mu \subset \lambda$. This follows immediately from the fact that $f_\lambda s^*_\mu(\lambda)/(n \downharpoonleft k) = \sum_\nu c^\lambda_{\mu\nu} f_\nu = \dim \lambda/\mu$, where $\dim \lambda/\mu$ is the number of standard numberings of the skew diagram λ/μ. This is a positive integer when $\mu \subset \lambda$ and zero otherwise. Indeed, when $\mu \subset \lambda$ it is clear that the matrix B with $\lambda_i - \mu_i$ down the diagonal satisfies $A + B = C$, where A and C are diagonal with spectra μ and λ, respectively. Thus this "two matrix" version of Horn's conjecture of the single marginal problem is essentially trivial, unlike its dual counterpart, the single marginal problem. However, note that the single marginal problem is also trivial, by Proposition IV.2, in the sense that the condition $\chi^{\lambda\mu}(q) > 0$ is always satisfied, unless one also specifies the dimension q of the traced-out subsystem.

V. GENERAL WERNER STATES.

We now drop the assumption that the state is symmetrical, and consider a general Werner state. First we characterise such states.

Proposition V.1 (Werner state characterisation)**.** *Any Werner state ρ can be written*

$$\rho = \sum_\lambda \sum_i r^i_\lambda P^i_{U_\lambda},$$

where r^i_λ are positive constants, and $P^i_{U_\lambda}$ are projectors onto unitary irreps U^i_λ.

Proof. If $\rho = \sum \gamma_i |a_i\rangle\langle a_i|$ is the eigenvalue decomposition of ρ, unitary invariance implies

$$\rho = \sum \gamma_i \mathbb{T}(|a_i\rangle\langle a_i|).$$

From the Schur-Weyl decomposition, we can write

$$|a_i\rangle = \sum_\lambda \gamma_{i,\lambda} |a_{i,\lambda}\rangle,$$

and Schur's lemma then tells us that

$$\mathbb{T}(|a_i\rangle\langle a_i|) = \sum_\lambda |\gamma_{i,\lambda}|^2 \mathbb{T}(|a_{i,\lambda}\rangle\langle a_{i,\lambda}|).$$

Let U^i_λ be the subspace generated by the set $\{U|a_{i,\lambda}\rangle \mid U \in \mathcal{U}(d)\}$. This is a unitary irrep, and $\mathbb{T}(|a_{i,\lambda}\rangle\langle a_{i,\lambda}|)$ is an intertwining operator from U^i_λ to itself, and hence by Schur's lemma is proportional to the projector $P^i_{U_\lambda}$. $\square$

Corollary V.2. *The number of (real) degrees of freedom of the set of Werner states on $(\mathbb{C}^d)^{\otimes n}$ is $d_W = \sum f^2_\lambda - 1$, where the sum is over λ in $Par(n, d)$.*

Proof. Another way of stating the result of the Proposition is that the λ-isotypic part of any Werner state is isomorphic to $\rho \otimes P_{U_\lambda}$, where ρ is any density matrix on V_λ. But ρ has f_λ real terms down the diagonal, with one constraint due to the sum of eigenvalues being 1, and there are $f_\lambda(f_\lambda - 1)$ real components in the non-diagonal terms above the diagonal, and those below the diagonal are the conjugates of those above. $\square$

We are now ready to prove the main theorem:

Theorem V.3 (General dual de Finetti theorem)**.** *If ρ is a Werner state on $(\mathbb{C}^p \otimes \mathbb{C}^q)^{\otimes n}$ and $q \geq n$, then*

$$||\mathsf{tr}_{\mathbb{C}^q}\rho - \frac{\mathcal{I}}{p^n}|| \leq 2 - 2\left(\frac{q-n+1}{q}\right)^n = \frac{2n(n-1)}{q} + O(n^4/q^2).$$

Proof. By Proposition V.1, it suffices to consider a state ρ that is a normalised projector onto a unitary irrep, i.e. a state of the form

$$\rho = P_{U_\lambda}/e_\lambda^{pq}. \tag{12}$$

Let $\{a_i\}$ and $\{b_j\}$ be bases for $\mathbb{C}^p$ and $\mathbb{C}^q$, respectively. We can define the Cartan subgroup of $\mathcal{U}(pq)$ as the set of matrices diagonal in the product basis $\{a_i \otimes b_j\}$. Let $\mathcal{F}$ denote the set of lexicographically ordered n-tuples of elements of this basis, which we write as $((i_1 j_1)\dots(i_n j_n))$; these define the weights of U_λ. Let $\mathcal{D}$ be the subset of $\mathcal{F}$ where the j indices are distinct; this set is non-empty because we are assuming $q \geq n$. The corresponding weight vectors are linear combinations of terms whose indices are permutations of those that occur in the weight, i.e. $((i_{\pi(1)}j_{\pi(1)})\dots(i_{\pi(n)}j_{\pi(n)}))$ for some permutation $\pi \in S_n$.

Let $U_\lambda^{\mathcal{D}}$ be the subspace of U_λ consisting of all the weight spaces for elements of $\mathcal{D}$. A permutation of the product basis $\{a_i \otimes b_j\}$, which can be regarded as an element of S_{pq}, induces a unitary map on $U_\lambda^{\mathcal{D}}$, and hence $P_{U_\lambda}^{\mathcal{D}}$, the projector on $U_\lambda^{\mathcal{D}}$, is invariant under such permutations. This implies that terms of the form

$$\left(|a_{i_{\pi(1)}}\rangle\langle a_{i_{\pi(1)}}| \otimes |b_{j_{\pi(1)}}\rangle\langle b_{j_{\pi(1)}}|\right) \otimes \dots \otimes \left(|a_{i_{\pi(n)}}\rangle\langle a_{i_{\pi(n)}}| \otimes |b_{j_{\pi(n)}}\rangle\langle b_{j_{\pi(n)}}|\right)$$

in $P_{U_\lambda}^{\mathcal{D}}$ all have the same coefficients, since any two such terms with different permuations π in S_n can be mapped into each other by an appropriate basis permutation in S_{pq}. Thus $tr_{\mathbb{C}^q} P_{U_\lambda}^{\mathcal{D}}$, i.e. the result of tracing out the $|b_j\rangle s$ from $P_{U_\lambda}^{\mathcal{D}}$, is a sum of terms

$$|a_{i_{\pi(1)}}\rangle\langle a_{i_{\pi(1)}}| \otimes \dots \otimes |a_{i_{\pi(n)}}\rangle\langle a_{i_{\pi(n)}}|,$$

for all $\pi \in S_n$, all terms having equal coefficients. Therefore $tr_{\mathbb{C}^q} P_{U_\lambda}^{\mathcal{D}}$ is proportional to the identity $\mathcal{I}$ on $(\mathbb{C}^p)^{\otimes n}$.

Now $U_\lambda^{\mathcal{D}}$ is the union of weight spaces, all of which are isomorphic and have dimension given by the Kostka number $K_{\lambda,(1^n)}$, which is f_λ (see [5, p. 56-57]). As there are p^n sets of possible i-indices in $\mathcal{D}$ and $\binom{q}{n}$ sets of distinct j-indices, $U_\lambda^{\mathcal{D}}$ has dimension $f_\lambda\binom{q}{n}p^n$. Thus,

$$tr_{\mathbb{C}^q} P_{U_\lambda}^{\mathcal{D}} = f_\lambda \binom{q}{n}\mathcal{I}.$$

From this and eq. (12),

$$tr_{\mathbb{C}^q}\rho = \frac{tr_{\mathbb{C}^q} P_{U_\lambda}}{e_\lambda^{pq}} = \frac{tr_{\mathbb{C}^q} P_{U_\lambda}^{\mathcal{D}}}{e_\lambda^{pq}} + A = \frac{f_\lambda\binom{q}{n}\mathcal{I}}{e_\lambda^{pq}} + A,$$

where A is a positive operator comes from tracing out the remaining weight subspaces in $P_{U_\lambda} - P_{U_\lambda}^{D}$. Thus, from the triangle inequality

$$||tr_{\mathbb{C}^q}\rho - \frac{\mathcal{I}}{p^n}|| \leq \left(1 - \frac{f_\lambda\binom{q}{n}p^n}{e_\lambda^{pq}}\right) + ||A|| = 2\left(1 - \frac{f_\lambda\binom{q}{n}p^n}{e_\lambda^{pq}}\right).$$

The remainder of the proof consists in finding a lower bound for $f_\lambda\binom{q}{n}/e_\lambda^{pq}$. To do this, we use the Weyl dimension formula for e_λ^{pq} and the hooklength formula for f_λ [5] to write

$$\frac{f_\lambda}{e_\lambda^{pq}} = \frac{n!(pq-1)!(pq-2)!\dots 1!}{(pq+\lambda_1-1)!(pq+\lambda_2-2)!\dots\lambda_d!}.$$

This ratio decreases when a box in the diagram λ is moved upwards, so it achieves its minimum for the diagram (n), giving

$$\frac{f_\lambda}{e_\lambda^{pq}} \geq \frac{n!(pq-1)!}{(pq+n-1)!}.$$

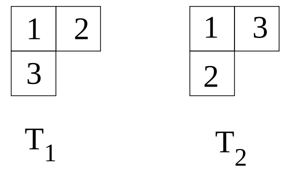

FIG. 1: The two tableaux for $(2,1)$; see Example V.4.

Combining this with the inequality

$$\frac{p(q-i+1)}{(pq+n-i)} \geq \frac{q-n+1}{q},$$

which holds for $1 \leq i \leq n$, one concludes

$$\frac{f_\lambda}{e_\lambda^{pq}}\binom{q}{n}p^n = \frac{q(q-1)\dots(q-n+1)p^n}{(pq+n-1)(pq+n-2)\dots pq} \geq \left(\frac{q-n+1}{q}\right)^n. \tag{13}$$

□

Example V.4. *The simplest diagram λ where V_λ is non-trivial is $(2,1)$. Here $f_\lambda = 2$, corresponding to the fact that there are two standard tableaux (numberings of λ that increase downwards and to the right), shown in Figure 1 as T_1 and T_2. Let $U_{\lambda,1}$ denote the unitary representation obtained by applying the Young projector [5] for the tableau T_1. The normalised projector $\rho = P_{U_{\lambda,1}}/e_\lambda^{pq}$ onto this representation is an example of a Werner state that is not symmetric. We explicitly calculate an approximation to the trace $tr_{\mathbb{C}^q}\rho$ of this state.*

As in the above proof, let $\{a_i\}$ and $\{b_i\}$ be bases for $\mathbb{C}^p$ and $\mathbb{C}^q$. Given (i_1,i_2,i_3) and distinct (j_1,j_2,j_3), let us write

$$|u_{xyz}\rangle = (|a_{i_x}\rangle \otimes |b_{j_x}\rangle) \otimes (|a_{i_y}\rangle \otimes |b_{j_y}\rangle) \otimes (|a_{i_z}\rangle \otimes |b_{j_z}\rangle),$$

where x,y,z is some permutation of $1,2,3$. Applying the Young projector to the $|u_{xyz}\rangle$ for all possible permutations of $1,2,3$ gives the set of vectors

$$\begin{aligned}
|\psi_1\rangle &= (|u_{123}\rangle + |u_{213}\rangle - |u_{321}\rangle - |u_{312}\rangle)/2, \\
|\psi_2\rangle &= (|u_{132}\rangle + |u_{312}\rangle - |u_{231}\rangle - |u_{213}\rangle)/2, \\
|\psi_3\rangle &= (|u_{321}\rangle + |u_{231}\rangle - |u_{123}\rangle - |u_{132}\rangle)/2,
\end{aligned}$$

These are linearly dependent, since $|\psi_1\rangle + |\psi_2\rangle + |\psi_3\rangle = 0$, and make the same angle with each other, since $\langle\psi_i|\psi_j\rangle = -1/2$ for all $i \neq j$. Thus the projector onto the 2D subspace they span is

$$\frac{2}{3}\left(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|\right).$$

Summing this expression over all (i_1,i_2,i_3) and distinct (j_1,j_2,j_3) gives the projector $P_{U^D_{\lambda,1}}$.

Observe that $P_{U^D_{\lambda,1}}$ is not symmetric; for instance $|u_{123}\rangle\langle u_{321}|$ occurs with coefficient $-1/3$, whereas $|u_{213}\rangle\langle u_{231}|$ has coefficient $1/6$. However, $|u_{xyz}\rangle\langle u_{xyz}|$ has the same coefficient, $1/3$, for all permutations x,y,z of $1,2,3$, and it is only these terms that contribute to the trace $tr_{\mathbb{C}^q}P_{U^D_{\lambda,1}}$. Summing over distinct indices (j_1,j_2,j_3) we therefore find

$$tr_{\mathbb{C}^q}P_{U^D_{\lambda,1}} = 3!\binom{q}{3}\left(\frac{1}{3}\right)\mathcal{I}.$$

The factor $3!$ here arises because, for distinct (i_1,i_2,i_3), there are $3!$ ways of combining them with a set of distinct (j_1,j_2,j_3); eg (i_1j_1, i_2j_2, i_3j_3), (i_1j_2, i_2j_1, i_3j_3), etc. When (i_1,i_2,i_3) are not distinct, there are fewer ways of combining them with (j_1,j_2,j_3), but on tracing out we regain the lost factor.

Since $e_\lambda^d = d(d-1)(d+1)/3$ for $\lambda = (2,1)$, we can write

$$tr_{\mathbb{C}^q}\frac{P_{U^D_{\lambda,1}}}{e_\lambda^{pq}} = \left[\frac{(q-1)(q-2)}{(q-1/p)(q+1/p)}\right]\frac{\mathcal{I}}{p^3}.$$

Let α denote the term in square brackets. Then

$$tr_{\mathbb{C}^q}\rho = \alpha\frac{\mathcal{I}}{p^3} + A,$$

and we see that $\alpha \to 1$ for $q \to \infty$.

To conclude this section, we look at the dual to Proposition V.1 and its corollary.

Proposition V.5 (Symmetric state characterisation)**.** *Any symmetric state ρ can be written*

$$\rho = \sum_\lambda \sum_i r^i_\lambda P^i_{V_\lambda},$$

where r^i_λ are positive constants, and $P^i_{V_\lambda}$ are projectors onto irreps V^i_λ of the symmetric group.

Corollary V.6. *The number of degrees of freedom of the set of symmetric states on $(\mathbb{C}^d)^{\otimes n}$ is $d_S = \sum(e^d_\lambda)^2 - 1$, where the sum is over λ in $Par(n,d)$.*

One might wonder if the standard deFinetti theorem could be proved by methods like those used for Theorem V.3. It seems that this is not possible, as the symmetric group representations have no analogue of the weight spaces that are essential for this proof.

VI. DISCUSSION

The de Finetti theorem and its dual seem very different in character. In the case of a symmetric Werner state ρ_λ, the standard de Finetti theorem tells us that $\mathsf{tr}_{n-k}\rho_\lambda$, the residual state when $n-k$ subsystems are traced out, can be approximated by the twirled power state $\mathbb{T}(\sigma^{\otimes k})$, where σ has spectrum $\bar{\lambda}$ (see Proposition III.3). If one carries out a measurement on $\mathbb{T}(\sigma^{\otimes k})$ of the projections onto the subspaces $U_\mu \otimes V_\mu$ in the Schur-Weyl decomposition of $(\mathbb{C}^d)^{\otimes k}$, the measured μ, normalised to $\bar{\mu}$, approximates $\bar{\lambda}$ [16, 17]. One will only get an accurate estimate if $k \gg d$; when this condition is satisfied, most of the information about the initial state is encoded in the traced-out state. By contrast, when part of each subsystem of a unitary-invariant state is traced out, the resulting state approximates a fully mixed state, which conveys no information about the initial state.

One might wonder whether this difference between the standard and dual de Finetti theorems is related to the number of parameters, d_S and d_W, needed to specify symmetric and Werner states, respectively. Is there a large reduction in d_W in tracing out $\mathbb{C}^q$ from each subsystem? If so, the loss of information about the initial state would be explained. However, this is not the case. In fact, for $p > n$, d_W is given by $\sum f^2_\lambda - 1$ over $\lambda \in Par(n,d)$ (Corollary V.2), and is the same for the whole system, where $d = pq$, and for the traced-out system where $d = p$. There is actually more of a reduction in the number of parameters with the standard de Finetti theorem, since d_S is given by $\sum_{\lambda\in Par(n,d)}(e^d_\lambda)^2 - 1$ (Corollary V.6), which does increase, though only polynomially, with n.

For the approximation to the fully mixed state to be close, the dimension q of the traced-out part of each subsystem must be large relative to $n(n-1)$, where n is the number of subsystems. Note that one does not require that p/q is small, where p is the dimension of the remaining part of each subsystem after tracing-out. The situation is therefore not directly analogous to the standard de Finetti theorem, where a good approximation requires that $(n-k)/n$, the ratio of the number of subsystems traced out to the total number of subsystems, be close to 1.

When $n = 1$, the bound in the dual de Finetti theorem is zero, which tells us that no tracing-out is needed; this just conveys the familiar fact that averaging the action of $\mathcal{U}(d)$ on a state on $\mathbb{C}^d$ gives the fully mixed state. One can ask which finite subsets S of $\mathcal{U}(d)$ have the property that the average $\sum_S U\rho U^\dagger/|S|$ gives a good approximation to the fully mixed state for any ρ, and it is known [18] that there are such sets with $|S| \approx d\log d$. The same question can be posed for $n > 1$, though now we expect to have to trace out part of each subsystem to get an approximation to the completely mixed state.

The dual de Finetti theorem has a certain resemblance to a theorem proved in [19]. This asserts that if H_E, the state space of the environment, is traced-out from a random state ρ on the product of the system and environment $H_S \otimes H_E$, then $\mathsf{tr}_E\rho$ is approximately a fully mixed state, the approximation improving as $\dim H_E/\dim H_S$ increases. (Actually the theorem holds more generally, for a state defined on an arbitrary subspace of $H_S \otimes H_E$.) This suggests that obtaining the fully mixed state after tracing out should be a property that holds for "almost all states", and not just for those with the special structure of Werner states. One might therefore hope to be able to extend the dual de Finetti theorem to a larger class of states (though mathematics abounds with propositions known to be almost always true, yet where specific instances are rather hard to find).

A natural application is to quantum secret-sharing: the theorem tells us that this can be achieved by splitting up the subsystems of a Werner state and giving them to two or more parties. With two parties, for instance, each can have half of each subsystem, though the dimension of each subsystem has to be large relative to n for this to work. Note that the procedure relies on the fact that p/q does not have to be small; we need to be able to regard both $\mathbb{C}^{\otimes q}$ and $\mathbb{C}^{\otimes p}$ as the traced-out part (and similarly for more than two parties).

Finally, one can ask whether the de Finetti theorem and its dual are facets of some more all-embracing version of the theorem.

VII. ACKNOWLEDGEMENTS

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