A de Finetti Representation Theorem for Quantum Process Tomography

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In quantum process tomography, it is possible to express the experimenter's prior information as a sequence of quantum operations, i.e., trace-preserving completely positive maps. In analogy to de Finetti's concept of exchangeability for probability distributions, we give a definition of exchangeability for sequences of quantum operations. We then state and prove a representation theorem for such exchangeable sequences. The theorem leads to a simple characterization of admissible priors for quantum process tomography and solves to a Bayesian's satisfaction the problem of an *unknown* quantum operation.

I. INTRODUCTION

In quantum process tomography [1, 2, 3], an experimenter lets an incompletely specified device act on a quantum system prepared in an input state of his choice, and then performs a measurement (also of his choice) on the output system. This procedure is repeated many times over, with possibly different input states and different measurements, in order to accumulate enough statistics to assign a quantum operation to the device. Here and throughout the paper, by a quantum operation we mean a trace-preserving completely positive linear map—the most general description for (unconditioned) quantum-state evolution allowed by the laws of quantum mechanics [4]. Quantum process tomography has been demonstrated experimentally in liquid state nuclear magnetic resonance [5, 6], and recently a number of optical experiments [7, 8, 9] have implemented entanglement-assisted quantum process tomography. The latter is a procedure that exploits the fact that quantum process tomography is equivalent to quantum state tomography in a larger state space [10, 11, 12, 13].

In the usual description of process tomography, it is assumed that the device performs the same *unknown* quantum operation Φ every time it is used, and an experimenter's prior information about the device is expressed via a probability density $p(\Phi)$ over all possible

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operations. What, however, is the operational meaning of an unknown quantum operation? When does the action of a device leave off from an initial input so that the next input can be sent through? In particular, what gives the right to suppose that a device does not have memory or, for instance, does not entangle the successive inputs passing through it? These questions boil down to the need to explore a single issue: What essential assumptions must be made so that quantum process tomography is a logically coherent notion?

In this paper, we address this issue with a uniqueness theorem based on (quantum) Bayesian methodology [14, 15, 16, 17, 18, 19]. What is called for is a method of posing quantum process tomography that never requires the invocation of the concept of an unknown quantum operation. This can be done by focussing upon the action of a single *known* quantum operation $\Phi^{(N)}$, which acts upon N nominal inputs. In particular, we identify conditions under which $\Phi^{(N)}$, $(N = 1, 2, \ldots)$, can be represented as

$$\Phi^{(N)} = \int d\Phi \; p(\Phi) \, \Phi^{\otimes N} \;, \tag{1}$$

for some probability density $p(\Phi)$, and where the integration extends over all single-system quantum operations Φ. With this theorem established, the conditions under which an experimenter can act *as if* his prior $\Phi^{(N)}$ corresponds to *ignorance* of a "true" but unknown quantum operation are made precise.

Our starting point is the closely aligned and similarly motivated de Finetti representation theorem for quantum states [16, 20, 21]. According to this theorem, a state $\rho^{(N)}$ of N systems can be written in the form

$$\rho^{(N)} = \int d\rho \; p(\rho) \, \rho^{\otimes N} \tag{2}$$

if and only if $\rho^{(N)}$ is an element of an *exchangeable sequence*. A quantum state $\rho^{(N)}$ of N systems is said to be a member of an exchangeable sequence if

(i) $\rho^{(k)}$ is symmetric, i.e., is invariant under permutations of the k systems on which it is defined, and

(ii) $\rho^{(k)} = \mathrm{tr}_{k+1}\rho^{(k+1)}$, where tr_{k+1} denotes the partial trace over the $(k+1)$th system.

In representation (2), $d\rho$ is a suitable measure on the density operator space, and $p(\rho) \geq 0$ is unique. The concept of exchangeability [22] was first introduced by Bruno de Finetti for sequences of probability distributions.

Here, we make use of the correspondence between quantum process tomography and quantum state tomography mentioned above to derive a de Finetti representation theorem for sequences of quantum operations. In Sec. II we define exchangeability for quantum operations and state the theorem. The proof is given in Sec. III. We close the paper with some concluding remarks that emphasize the quantum foundational character of our result.

II. THE THEOREM

In this paper, we restrict our attention to devices for which the input and output have the same Hilbert space dimension, D. In the following, $\mathcal{H}_D$ denotes a D-dimensional Hilbert space, $\mathcal{H}_D^{\otimes N} = \mathcal{H}_D \otimes \cdots \otimes \mathcal{H}_D$ denotes its N-fold tensor product, and $\mathcal{L}(\mathcal{V})$ denotes the space of linear operators on a linear space $\mathcal{V}$. The set of density operators for a D-dimensional quantum system is a convex subset of $\mathcal{L}(\mathcal{H}_D)$.

The action of a device on N nominal inputs systems is then described by a trace-preserving completely positive map

$$\Phi^{(N)} : \mathcal{L}(\mathcal{H}_D^{\otimes N}) \longrightarrow \mathcal{L}(\mathcal{H}_D^{\otimes N}) \,, \tag{3}$$

which maps the state of the N input systems to the state of the N output systems. We will say, in analogy to the definition of exchangeability for quantum states, that a quantum operation $\Phi^{(N)}$ is *exchangeable* if it is a member of an exchangeable sequence of quantum operations.

To define exchangeability for a sequence of quantum operations in a natural way, we reduce the properties of symmetry and extendibility for sequences of operations to the corresponding properties for sequences of states. In the following, we will use bold letters to denote vectors of indices, e.g. $\mathbf{j} = (j_1, \ldots, j_N)$. We will use π to denote a permutation of the set $\{1, \ldots, N\}$, where the cardinality N will depend on the context. The action of the permutation π on the vector $\mathbf{j}$ is defined by $\pi\mathbf{j} = (j_{\pi(1)}, \ldots, j_{\pi(N)})$.

Any N-system density operator $\rho^{(N)}$ can be expanded in the form

$$\rho^{(N)} = \sum_{\mathbf{j},\mathbf{l}} r^{(N)}_{\mathbf{j},\mathbf{l}} \bigotimes_{i=1}^{N} |j_i^{Q_i}\rangle\langle l_i^{Q_i}| \equiv \sum_{\mathbf{j},\mathbf{l}} r^{(N)}_{\mathbf{j},\mathbf{l}} |j_1^{Q_1}\rangle\langle l_1^{Q_1}| \otimes \cdots \otimes |j_N^{Q_N}\rangle\langle l_N^{Q_N}| \,, \tag{4}$$

where $\{|1^{Q_i}\rangle, \ldots, |D^{Q_i}\rangle\}$ denotes an orthonormal basis for the Hilbert space $\mathcal{H}_D$ of the ith system, and $r^{(N)}_{\mathbf{j},\mathbf{l}}$ are the matrix elements of $\rho^{(N)}$ in the tensor product basis. We define the action of the permutation π on the state $\rho^{(N)}$ by

$$\pi\rho^{(N)} = \sum_{\mathbf{j},\mathbf{l}} r^{(N)}_{\pi\mathbf{j},\pi\mathbf{l}} \bigotimes_{i=1}^{N} |j_i^{Q_i}\rangle\langle l_i^{Q_i}| = \sum_{\mathbf{j},\mathbf{l}} r^{(N)}_{\mathbf{j},\mathbf{l}} \bigotimes_{i=1}^{N} |j_{\pi^{-1}(i)}^{Q_i}\rangle\langle l_{\pi^{-1}(i)}^{Q_i}| \,. \tag{5}$$

With this notation, we can make the following definition.

Definition 1 *A sequence of quantum operations, $\Phi^{(k)} : \mathcal{L}(\mathcal{H}_D^{\otimes k}) \to \mathcal{L}(\mathcal{H}_D^{\otimes k})$, is exchangeable if, for $k = 1, 2, \ldots$,*

(i) $\Phi^{(k)}$ is symmetric, i.e.,

$$\Phi^{(k)}(\rho^{(k)}) = \pi\Big(\Phi^{(k)}(\pi^{-1}\rho^{(k)})\Big) \tag{6}$$

for any permutation π of the set $\{1, \ldots, k\}$ and for any density operator $\rho^{(k)} \in \mathcal{L}(\mathcal{H}_D^{\otimes k})$, and

(ii) $\Phi^{(k)}$ is extendible, i.e.,

$$\Phi^{(k)}(\mathrm{tr}_{k+1}\rho^{(k+1)}) = \mathrm{tr}_{k+1}\Big(\Phi^{(k+1)}(\rho^{(k+1)})\Big) \tag{7}$$

for any state $\rho^{(k+1)}$.

In words, these conditions amount to the following. Condition (i) is equivalent to the requirement that the quantum operation $\Phi^{(k)}$ commutes with any permutation operator π acting on the states $\rho^{(k)}$: It does not matter what order we send our systems through the device; as long as we rearrange them at the end into the original order, the resulting evolution will be the same. Condition (ii) says that it does not matter if we consider a larger map $\Phi^{(N+1)}$ acting on a larger collection of systems (possibly entangled), or a smaller $\Phi^{(N)}$ on some subset of those systems: The upshot of the evolution will be the same for the relevant systems.

We are now in a position to formulate the de Finetti representation theorem for quantum operations.

Theorem 1 *A quantum operation $\Phi^{(N)} : \mathcal{L}(\mathcal{H}_D^{\otimes N}) \to \mathcal{L}(\mathcal{H}_D^{\otimes N})$ is an element of an exchangeable sequence if and only if it can be written in the form*

$$\Phi^{(N)} = \int d\Phi \, p(\Phi)\, \Phi^{\otimes N} \quad \textit{for all } N, \tag{8}$$

where the integral ranges over all single-shot quantum operations $\Phi : \mathcal{L}(\mathcal{H}_D) \to \mathcal{L}(\mathcal{H}_D)$, $d\Phi$ is a suitable measure on the space of quantum operations, and the probability density $p(\Phi) \geq 0$ is unique. The tensor product $\Phi^{\otimes N}$ is defined by $\Phi^{\otimes N}(\rho_1 \otimes \cdots \otimes \rho_N) = \Phi(\rho_1) \otimes \cdots \otimes \Phi(\rho_N)$ for all $\rho_1, \ldots, \rho_N$ and by linear extension for arbitrary arguments.

Just as with the original quantum de Finetti theorem [16, 20], this result allows a certain latitude in how quantum process tomography can be described. One is free to use the language of an unknown quantum operation if the condition of exchangeability is met by one's prior $\Phi^{(N)}$ but it is not required: For the (quantum) Bayesian statistician the *known* quantum operation $\Phi^{(N)}$ is the more fundamental object.

III. PROOF

Let $\Phi^{(N)}$, $N = 1, 2, \ldots$, be an exchangeable sequence of quantum operations. $\Phi^{(N)}$ can be characterized in terms of its action on the elements of a basis of $\mathcal{L}(\mathcal{H}_D^{\otimes N})$ as follows.

$$\Phi^{(N)}\Big(\bigotimes_{i=1}^{N} |j_i^{Q_i}\rangle\langle k_i^{Q_i}| \Big) = \sum_{\mathbf{l},\mathbf{m}} S^{(N)}_{\mathbf{l},\mathbf{j},\mathbf{m},\mathbf{k}} \bigotimes_{i=1}^{N} |l_i^{Q_i}\rangle\langle m_i^{Q_i}| \; . \tag{9}$$

The coefficients $S^{(N)}_{\mathbf{l},\mathbf{j},\mathbf{m},\mathbf{k}}$ specify $\Phi^{(N)}$ uniquely. It follows from a construction due to Jamiołkowski [23] that the $S^{(N)}_{\mathbf{l},\mathbf{j},\mathbf{m},\mathbf{k}}$ can be regarded as the matrix elements of a density operator on D^{2N}-dimensional Hilbert space $\mathcal{H}_{D^2}^{\otimes N}$. This can be seen as follows. Let

$$|\Psi\rangle = \frac{1}{\sqrt{D}} \sum_{k=1}^{D} |k^{R_i}\rangle|k^{Q_i}\rangle \in \mathcal{H}_D \otimes \mathcal{H}_D = \mathcal{H}_{D^2} \tag{10}$$

be a maximally entangled state in $\mathcal{H}_{D^2}$, where the $|k^{R_i}\rangle$ ($k = 1, \ldots, D$) form orthonormal bases for the ancillary systems labelled R_i ($i = 1, \ldots, N$). The corresponding density operator is

$$|\Psi\rangle\langle\Psi| = \frac{1}{D} \sum_{j,k} |j^{R_i}\rangle\langle k^{R_i}| \otimes |j^{Q_i}\rangle\langle k^{Q_i}| \in \mathcal{L}(\mathcal{H}_{D^2}) \; . \tag{11}$$

Similarly, we define a map, J, from the set of quantum operations on $\mathcal{H}_D^{\otimes N}$ to the set of density operators on $\mathcal{H}_{D^2}^{\otimes N}$ by

$$J(\Phi^{(N)}) \;\equiv\; \Big(I^{(N)} \otimes \Phi^{(N)}\Big)\Big((|\Psi\rangle\langle\Psi|)^{\otimes N}\Big)$$

$$= \frac{1}{D^N}\Big(I^{(N)}\otimes\Phi^{(N)}\Big)\Big(\sum_{\mathbf{j},\mathbf{k}}\bigotimes_{i=1}^{N}(|j_i^{R_i}\rangle\langle k_i^{R_i}|\otimes|j_i^{Q_i}\rangle\langle k_i^{Q_i}|)\Big)$$
$$= \frac{1}{D^N}\sum_{\mathbf{l},\mathbf{j},\mathbf{m},\mathbf{k}} S^{(N)}_{\mathbf{l},\mathbf{j},\mathbf{m},\mathbf{k}}\bigotimes_{i=1}^{N}(|j_i^{R_i}\rangle\langle k_i^{R_i}|\otimes|l_i^{Q_i}\rangle\langle m_i^{Q_i}|)\ . \qquad (12)$$

In this definition, $I^{(N)}$ denotes the identity operation acting on the ancillary systems $R_1,\ldots,R_N$. The map J is injective, i.e. $J(\Phi_1^{(N)})=J(\Phi_2^{(N)})$ if and only if $\Phi_1^{(N)}=\Phi_2^{(N)}$.

The first stage of the proof of the de Finetti theorem for operations is to show that the density operators $J(\Phi^{(N)})$, $N=1,2,\ldots$, form an exchangeable sequence when regarded as N-system states, with R_i and Q_i jointly forming the ith system. To do this, we first show that $J(\Phi^{(N)})$ is symmetric, i.e., invariant under an arbitrary permutation π of the N systems.

Note that since the density operators $\rho^{(N)}$ actually span the whole vector space $\mathcal{L}(\mathcal{H}_D^{\otimes N})$, enforcing Definition 1 above amounts to identifying the linear maps on the left- and right-hand sides of Eqs. (6) and (7). I.e.,

$$\Phi^{(k)}=\pi\circ\Phi^{(k)}\circ\pi^{-1} \qquad (13)$$

and

$$\Phi^{(k)}\circ\mathrm{tr}_{k+1}=\mathrm{tr}_{k+1}\circ\Phi^{(k+1)} \qquad (14)$$

Thus in much that we do it suffices to consider the action of these maps on an arbitrary basis state $E^{(N)}=\bigotimes_{i=1}^{N}|j_i^{Q_i}\rangle\langle k_i^{Q_i}|$ for arbitrary $\mathbf{j}$ and $\mathbf{k}$. In particular,

$$\pi\Big(\Phi^{(N)}(\pi^{-1}E^{(N)})\Big) = \pi\Big(\Phi^{(N)}\Big(\bigotimes_{i=1}^{N}|j_{\pi(i)}^{Q_i}\rangle\langle k_{\pi(i)}^{Q_i}|\Big)\Big)$$
$$= \pi\sum_{\mathbf{l},\mathbf{m}} S^{(N)}_{\mathbf{l},\pi\mathbf{j},\mathbf{m},\pi\mathbf{k}}\bigotimes_{i=1}^{N}|l_i^{Q_i}\rangle\langle m_i^{Q_i}|$$
$$= \sum_{\mathbf{l},\mathbf{m}} S^{(N)}_{\pi\mathbf{l},\pi\mathbf{j},\pi\mathbf{m},\pi\mathbf{k}}\bigotimes_{i=1}^{N}|l_i^{Q_i}\rangle\langle m_i^{Q_i}|\ . \qquad (15)$$

Assuming Eq. (6), i.e., symmetry of $\Phi^{(N)}$, for all $\mathbf{j}$ and $\mathbf{k}$, it follows that

$$S^{(N)}_{\pi\mathbf{l},\pi\mathbf{j},\pi\mathbf{m},\pi\mathbf{k}}=S^{(N)}_{\mathbf{l},\mathbf{j},\mathbf{m},\mathbf{k}} \qquad (16)$$

for all $\mathbf{l},\mathbf{j},\mathbf{m},\mathbf{k}$, which, using Eq. (12), implies that

$$\pi(J(\Phi^{(N)}))=J(\Phi^{(N)})\ , \qquad (17)$$

i.e., symmetry of $J(\Phi^{(N)})$.

To prove extendibility of $J(\Phi^{(N)})$, we introduce the following notation for partial traces: we denote by tr^R_{N+1} the partial trace over the subsystem R_{N+1}, and by tr^Q_{N+1} the partial trace over the subsystem Q_{N+1}. In this notation, we need to show that $\mathrm{tr}^R_{N+1}\mathrm{tr}^Q_{N+1}J(\Phi^{(N+1)})=J(\Phi^{(N)})$. Using Eqs. (7) and (12),

$$\mathrm{tr}^R_{N+1}\mathrm{tr}^Q_{N+1}J(\Phi^{(N+1)})$$
$$=\mathrm{tr}^R_{N+1}\mathrm{tr}^Q_{N+1}\frac{1}{D^{N+1}}\Big(I^{(N+1)}\otimes\Phi^{(N+1)}\Big)\Big(\sum_{\mathbf{j},j_{N+1},\mathbf{k},k_{N+1}}\bigotimes_{i=1}^{N+1}(|j_i^{R_i}\rangle\langle k_i^{R_i}|\otimes|j_i^{Q_i}\rangle\langle k_i^{Q_i}|)\Big)$$

$$
\begin{aligned}
&= \mathrm{tr}^Q_{N+1} \frac{1}{D^{N+1}} \Big(I^{(N)} \otimes \Phi^{(N+1)} \Big) \Big(\sum_{\mathbf{j},\mathbf{k},k_{N+1}} \bigotimes_{i=1}^{N} (|j_i^{R_i}\rangle\langle k_i^{R_i}| \otimes |j_i^{Q_i}\rangle\langle k_i^{Q_i}|) \otimes |k_{N+1}^{Q_{N+1}}\rangle\langle k_{N+1}^{Q_{N+1}}| \Big) \\
&= \frac{1}{D^{N+1}} \sum_{\mathbf{j},\mathbf{k},k_{N+1}} \Big(\bigotimes_{i=1}^{N} (|j_i^{R_i}\rangle\langle k_i^{R_i}| \Big) \otimes \mathrm{tr}^Q_{N+1} \Phi^{(N+1)} \Big(\bigotimes_{l=1}^{N} |j_l^{Q_i}\rangle\langle k_l^{Q_i}| \otimes |k_{N+1}^{Q_{N+1}}\rangle\langle k_{N+1}^{Q_{N+1}}| \Big) \\
&= \frac{1}{D^{N+1}} \sum_{\mathbf{j},\mathbf{k},k_{N+1}} \Big(\bigotimes_{i=1}^{N} (|j_i^{R_i}\rangle\langle k_i^{R_i}| \Big) \otimes \Phi^{(N)} \Big(\bigotimes_{l=1}^{N} |j_l^{Q_i}\rangle\langle k_l^{Q_i}| \Big) \\
&= \frac{1}{D^{N+1}} \Big(I^{(N)} \otimes \Phi^{(N)} \Big) \Big(\sum_{\mathbf{j},\mathbf{k},k_{N+1}} \bigotimes_{i=1}^{N} (|j_i^{R_i}\rangle\langle k_i^{R_i}| \otimes |j_i^{Q_i}\rangle\langle k_i^{Q_i}|) \Big) \\
&= \frac{1}{D^{N}} \Big(I^{(N)} \otimes \Phi^{(N)} \Big) \Big(\sum_{\mathbf{j},\mathbf{k}} \bigotimes_{i=1}^{N} (|j_i^{R_i}\rangle\langle k_i^{R_i}| \otimes |j_i^{Q_i}\rangle\langle k_i^{Q_i}|) \Big) \\
&= J(\Phi^{(N)}) \ . \qquad (18)
\end{aligned}
$$

We have thus shown that $J(\Phi^{(N)})$, $N = 1, 2, \ldots$, form an exchangeable sequence. According to the quantum de Finetti theorem for density operators [see Eq. (2)], we can write

$$
J(\Phi^{(N)}) = \int d\rho \, p(\rho) \, \rho^{\otimes N} \ , \qquad (19)
$$

where $p(\rho) \geq 0$ is unique, and $\int d\rho \, p(\rho) = 1$. With the parameterization

$$
\rho = \frac{1}{D} \sum_{l,j,m,k} S^{(1)}_{l,j,m,k} |j^R\rangle\langle k^R| \otimes |l^Q\rangle\langle m^Q| \ , \qquad (20)
$$

Eq. (19) takes the form

$$
\begin{aligned}
J(\Phi^{(N)}) &= \frac{1}{D^N} \int_{\mathcal{D}} dS \, p(S) \, \Big(\sum_{l,j,m,k} S^{(1)}_{l,j,m,k} |j^R\rangle\langle k^R| \otimes |l^Q\rangle\langle m^Q| \Big)^{\otimes N} \\
&= \frac{1}{D^N} \int_{\mathcal{D}} dS \, p(S) \, \bigotimes_{i=1}^{N} \sum_{l_i,j_i,m_i,k_i} S^{(1)}_{l_i,j_i,m_i,k_i} |j_i^{R_i}\rangle\langle k_i^{R_i}| \otimes |l_i^{Q_i}\rangle\langle m_i^{Q_i}| \\
&= \frac{1}{D^N} \sum_{\mathbf{l},\mathbf{j},\mathbf{m},\mathbf{k}} \int_{\mathcal{D}} dS \, p(S) \, \bigotimes_{i=1}^{N} S^{(1)}_{l_i,j_i,m_i,k_i} |j_i^{R_i}\rangle\langle k_i^{R_i}| \otimes |l_i^{Q_i}\rangle\langle m_i^{Q_i}| \ , \qquad (21)
\end{aligned}
$$

where the integration variable is a vector with D^4 components, $S = (S^{(1)}_{1,1,1,1}, \ldots, S^{(1)}_{D,D,D,D})$, and where the integration domain, $\mathcal{D}$, is the set of all S that represent matrix elements of a density operator. The function $p(S)$ is unique, $p(S) \geq 0$, and $\int_{\mathcal{D}} dS \, p(S) = 1$. Notice the slight abuse of notation in the first line of Eq. (21), where the superscripts R and Q label the entire sequences of systems $R_1, \ldots, R_N$ and $Q_1, \ldots, Q_N$, respectively.

Comparing Eq. (21) with Eq. (12), we can express the coefficients $S^{(N)}_{\mathbf{l},\mathbf{j},\mathbf{m},\mathbf{k}}$ specifying the quantum operation $\Phi^{(N)}$ [see Eq. (9)] in terms of the integral above:

$$
S^{(N)}_{\mathbf{l},\mathbf{j},\mathbf{m},\mathbf{k}} = \int_{\mathcal{D}} dS \, p(S) \, \prod_{i=1}^{N} S^{(1)}_{l_i,j_i,m_i,k_i} \ . \qquad (22)
$$

Hence, for any $\mathbf{j}$ and $\mathbf{k}$,

$$
\Phi^{(N)} \Big(\bigotimes_{i=1}^{N} |j_i^{Q_i}\rangle\langle k_i^{Q_i}| \Big) = \sum_{\mathbf{l},\mathbf{m}} \int_{\mathcal{D}} dS \, p(S) \, \Big(\prod_{i=1}^{N} S^{(1)}_{l_i,j_i,m_i,k_i} \Big) \bigotimes_{i=1}^{N} |l_i^{Q_i}\rangle\langle m_i^{Q_i}|
$$

$$= \int_{\mathcal{D}} dS\, p(S) \bigotimes_{i=1}^{N} \sum_{l_i,m_i} S^{(1)}_{l_i,j_i,m_i,k_i} |l_i^{Q_i}\rangle\langle m_i^{Q_i}| \; . \tag{23}$$

The D^4 coefficients, $S^{(1)}_{l,j,m,k}$, of the vector S define a single-system map, Φ_S, via

$$\Phi_S(|j^Q\rangle\langle k^Q|) \equiv \sum_{l,m} S^{(1)}_{l,j,m,k} |l^Q\rangle\langle m^Q| \quad (j,k = 1,\ldots,D) \; . \tag{24}$$

Hence

$$\begin{aligned} \Phi^{(N)}\Big(\bigotimes_{i=1}^{N} |j_i^{Q_i}\rangle\langle k_i^{Q_i}|\Big) &= \int_{\mathcal{D}} dS\, p(S) \bigotimes_{i=1}^{N} \Phi_S\Big(|j_i^{Q_i}\rangle\langle k_i^{Q_i}|\Big) \\ &= \int_{\mathcal{D}} dS\, p(S)\, \Phi_S^{\otimes N}\Big(\bigotimes_{i=1}^{N} |j_i^{Q_i}\rangle\langle k_i^{Q_i}|\Big) \; . \end{aligned} \tag{25}$$

Since this equality holds for arbitrary $\mathbf{j}$ and $\mathbf{k}$, it implies the representation

$$\Phi^{(N)} = \int_{\mathcal{D}} dS\, p(S)\, \Phi_S^{\otimes N} \; . \tag{26}$$

For all $S \in \mathcal{D}$, the map Φ_S is completely positive. This can be seen by considering

$$J(\Phi_S) = (I \otimes \Phi_S)(|\Psi\rangle\langle\Psi|) = \frac{1}{D} \sum_{l,j,m,k} S^{(1)}_{l,j,m,k} |j^R\rangle\langle k^R| \otimes |l^Q\rangle\langle m^Q| \; ,$$

which, by definition of $\mathcal{D}$, is a density operator and therefore positive. It follows from a theorem by Choi [24] that Φ_S is completely positive.

To complete the proof, we will now show that $p(S) = 0$ almost everywhere (a.e.) unless Φ_S is trace-preserving, i.e., a quantum operation. More precisely, we show that if $U \in \mathcal{D}$ is such that Φ_U is not trace-preserving, then there exists an open ball B containing U such that $p(S) = 0$ (a.e.) in $B \cap \mathcal{D}$.

For $\delta > 0$ and $U \in \mathcal{D}$, we define $B_\delta(U)$ to be the set of all S such that $|S - U| < \delta$, i.e., $B_\delta(U)$ is the open ball of radius δ centered at U. Furthermore, we define $\bar{B}_\delta(U) = B_\delta(U) \cap \mathcal{D}$.

Let $U \in \mathcal{D}$ be such that Φ_U is not trace-preserving, i.e., there exists a density operator ρ for which $\mathrm{tr}[\Phi_U(\rho)] \neq 1$. We distinguish two cases.

Case (i): $\mathrm{tr}[\Phi_U(\rho)] = 1 + \epsilon$, where $\epsilon > 0$. Since $\mathrm{tr}[\Phi_S(\rho)]$ is a linear and therefore continuous function of the vector S, there exists $\delta > 0$ such that

$$\Big|\mathrm{tr}[\Phi_S(\rho)] - \mathrm{tr}[\Phi_U(\rho)]\Big| < \epsilon/2 \tag{27}$$

whenever $S \in B_\delta(U)$. For $S \in \bar{B}_\delta(U)$,

$$\mathrm{tr}[\Phi_S(\rho)] > 1 + \epsilon - \epsilon/2 = 1 + \epsilon/2 \; . \tag{28}$$

Therefore

$$\begin{aligned} \mathrm{tr}[\Phi^{(N)}(\rho^{\otimes N})] &= \mathrm{tr}\left[\int_{\mathcal{D}} dS\, p(S)\, \Phi_S^{\otimes N}(\rho^{\otimes N})\right] \\ &= \int_{\mathcal{D}} dS\, p(S)\, \big(\mathrm{tr}[\Phi_S(\rho)]\big)^N \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathcal{D}\backslash\bar{B}_\delta(U)} dS\, p(S)\ \left(\mathrm{tr}[\Phi_S(\rho)]\right)^N + \int_{\bar{B}_\delta(U)} dS\, p(S)\ \left(\mathrm{tr}[\Phi_S(\rho)]\right)^N \\
&\geq \int_{\bar{B}_\delta(U)} dS\, p(S)\ \left(\mathrm{tr}[\Phi_S(\rho)]\right)^N \\
&> (1+\epsilon/2)^N \int_{\bar{B}_\delta(U)} dS\, p(S)\ . \qquad (29)
\end{aligned}
$$

Unless $\int_{\bar{B}_\delta(U)} dS\, p(S) = 0$, there exists N such that $\mathrm{tr}[\Phi^{(N)}(\rho^{\otimes N})] > 1$, which contradicts the assumption that $\Phi^{(N)}$ is trace-preserving. Hence $p(S) = 0$ (a.e.) in $\bar{B}_\delta(U)$.

Case (ii): $\mathrm{tr}[\Phi_U(\rho)] = 1 - \epsilon$, where $0 < \epsilon \leq 1$. Because of continuity, there exists $\delta > 0$ such that

$$\Big|\mathrm{tr}[\Phi_S(\rho)] - \mathrm{tr}[\Phi_U(\rho)]\Big| < \epsilon/2 \qquad (30)$$

whenever $S \in B_\delta(U)$. Hence, for $S \in \bar{B}_\delta(U)$,

$$\mathrm{tr}[\Phi_S(\rho)] < 1 - \epsilon + \epsilon/2 = 1 - \epsilon/2\ . \qquad (31)$$

Now assume that $\int_{\bar{B}_\delta(U)} dS\, p(S) = \eta > 0$. Then, letting $N = 1$,

$$
\begin{aligned}
1 = \mathrm{tr}[\Phi^{(1)}(\rho)] &= \mathrm{tr}\left[\int_{\mathcal{D}} dS\, p(S)\ \Phi_S(\rho)\right] \\
&= \int_{\mathcal{D}\backslash\bar{B}_\delta(U)} dS\, p(S)\ \mathrm{tr}[\Phi_S(\rho)] + \int_{\bar{B}_\delta(U)} dS\, p(S)\ \mathrm{tr}[\Phi_S(\rho)] \\
&< \int_{\mathcal{D}\backslash\bar{B}_\delta(U)} dS\, p(S)\ \mathrm{tr}[\Phi_S(\rho)] + \eta(1-\epsilon/2)\ , \qquad (32)
\end{aligned}
$$

which implies that

$$\int_{\mathcal{D}\backslash\bar{B}_\delta(U)} dS\, p(S)\ \mathrm{tr}[\Phi_S(\rho)] > 1 - \eta + \eta\epsilon/2 > 1 - \eta\ . \qquad (33)$$

Since

$$\int_{\mathcal{D}\backslash\bar{B}_\delta(U)} dS\, p(S) = 1 - \eta\ , \qquad (34)$$

it follows that there exist $\zeta > 0$ and a point $V \in \mathcal{D}\backslash\bar{B}_\delta(U)$ such that $\mathrm{tr}[\Phi_V(\rho)] > 1$ and

$$\int_{\bar{B}_\xi(V)} dS\, p(S) > 0 \quad \text{for all } \xi \leq \zeta\ . \qquad (35)$$

We are thus back to case (i) above. Repeating the argument of case (i) one can show that this contradicts the assumption that $\Phi^{(N)}$ is trace preserving for large N. It follows that $\eta = 0$, i.e., $p(S) = 0$ (a.e.) in $\bar{B}_\delta(U)$. This concludes the proof of the de Finetti theorem for quantum operations.

IV. CONCLUDING REMARKS

What we have proven here is a representation theorem. It shows us when an experimenter is warranted to think of his (prior) *known* quantum operation assignment as built out of a lack of knowledge of a "true" but *unknown* one. In that way, the theorem has the same kind of attraction as the previous quantum de Finetti theorem for quantum states [16, 20, 21].

In particular for an information-based interpretation of quantum mechanics such as the one being developed in Refs. [15, 16, 17], it may be a necessary ingredient for its very consistency. In Refs. [17, 25], it has been argued strenuously that quantum operations should be considered of essentially the same physical meaning and status as quantum states themselves: They are Bayesian expressions of an experimenter's judgment. This could be captured in the slogan "a quantum operation is really a quantum state in disguise." In other words, the Choi representation theorem [24] is not just a mathematical nicety, but is instead of deep physical significance.

Therefore, just as an unknown quantum state is an oxymoron in an information-based interpretation of quantum mechanics, so should be an unknown quantum operation. In the case of quantum states, the conundrum is solved by the existence of a de Finetti theorem for quantum tomography. Here we have shown that the conundrum in quantum process tomography can be solved in almost the same way. One might reject the arguments leading to the slogan that a quantum operation is a quantum state in disguise [26], but then one should be curious about the nice fit of the formalism to the philosophy.

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